Gems of TCS

VC Dimension

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- Goal: given training set, select h that approximates c well

Generalization Error

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For hypothesis h, target concept c, and target distribution D:

$$R(h) = \Pr_{x \sim D}[h(x) \neq c(x)] \, .$$

PAC Learning

PAC (Probably Approximately Correct)

Concept class C is PAC-learnable if there exists learning algorithm s.t.

- for all $c \in C, \varepsilon > 0, \delta > 0$, all distributions D:

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- for random samples of size

$$m \leq \mathrm{poly}(1/\varepsilon, 1/\delta)\,.$$

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A set of m instances/examples $S \in X^m$ is shattered if all 2^m are realizable by hypotheses from C.

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- To prove that VC dimension of C is d we need to
  - Show a set of d examples that can be shattered by C
  - Prove that every set of $d + 1$ examples cannot be shattered by C

Intervals on the Line

Half-planes

Half-spaces

Axis-aligned Rectangles

Convex Polygons

Fundamental Theorem

The Fundamental Theorem of Statistical Learning Theory

- If C has finite VC dimension, then C is PAC-learnable.
- If C has infinite VC dimension, then C is not PAC-learnable.