Gems of TCS

Easy and Hard Problems

Sasha Golovnev

January 26, 2020

Theoretical Computer Science

Theoretical Computer Science

$P \implies Q$

Mathematical logic

Theoretical Computer Science

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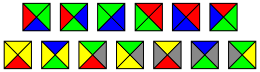
Mathematical logic



Computability theory

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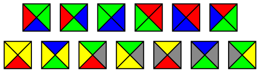

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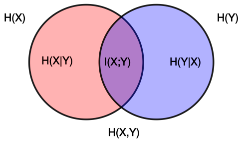

Information theory

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Mathematical
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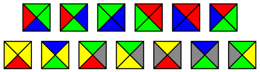

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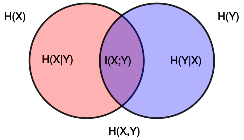

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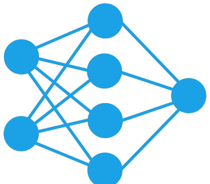

Learning,
neural nets

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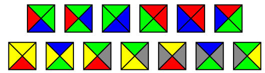

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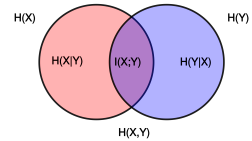

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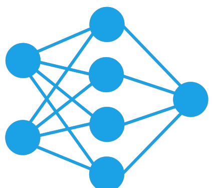

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$P = NP?$

Computational complexity

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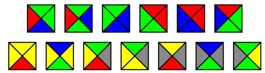

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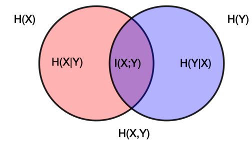

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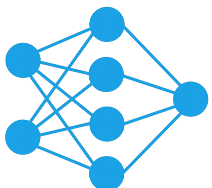

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Cryptography

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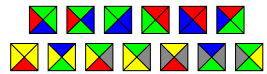

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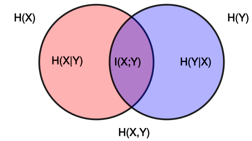

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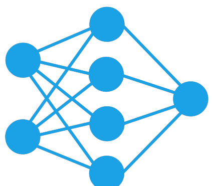

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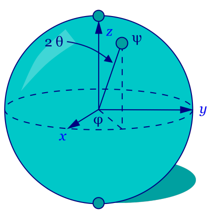

Quantum Algorithms

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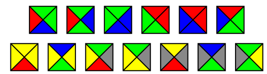

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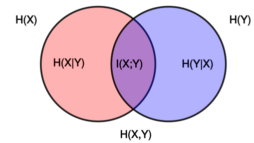

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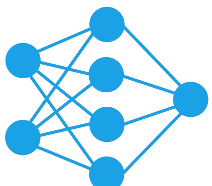

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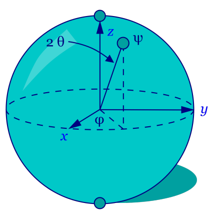

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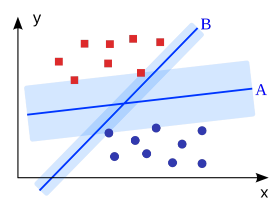

Machine learning

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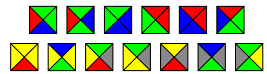

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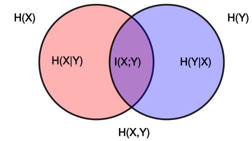

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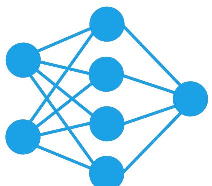

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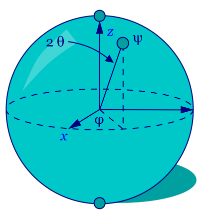

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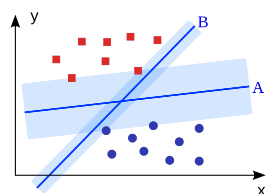

Machine learning

Data Science

This Course

- Theoretical/Mathematical viewpoint

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- Theoretical/Mathematical viewpoint
- Topic overview

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- Theoretical/Mathematical viewpoint
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    - Algorithms

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    - Learning

Administrative Info

- Classes: TR 9:30am–10:45am, Zoom

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- email: `alexgolovnev+gems@gmail.com`

Course Begins

- Running time of an algorithm

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  - $100n^2$ vs $n^3/10$

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- Complexity class **P** = problems solvable in time poly(n)

Course Begins

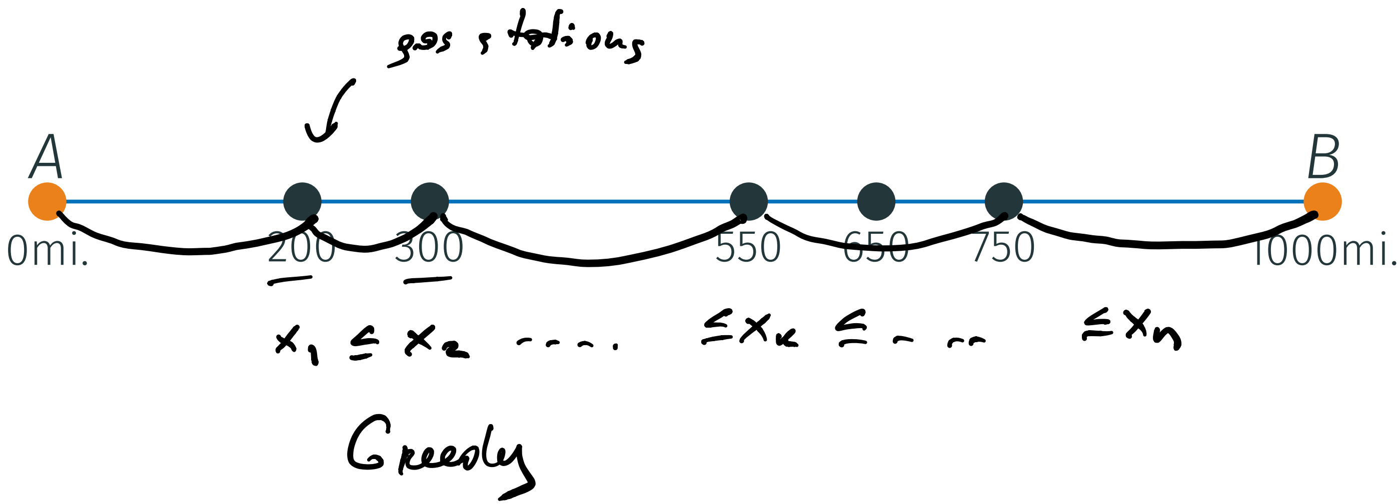
- Running time of an algorithm
  - $100n^2$ vs $n^3/10$
  - $100n^2$ vs $2^n/100$
- Complexity class **P** easy
- Complexity class **NP**-hard hard

Car Fueling

Car Fueling

Distance with full tank 300 mi.

Minimize the number of stops at gas stations

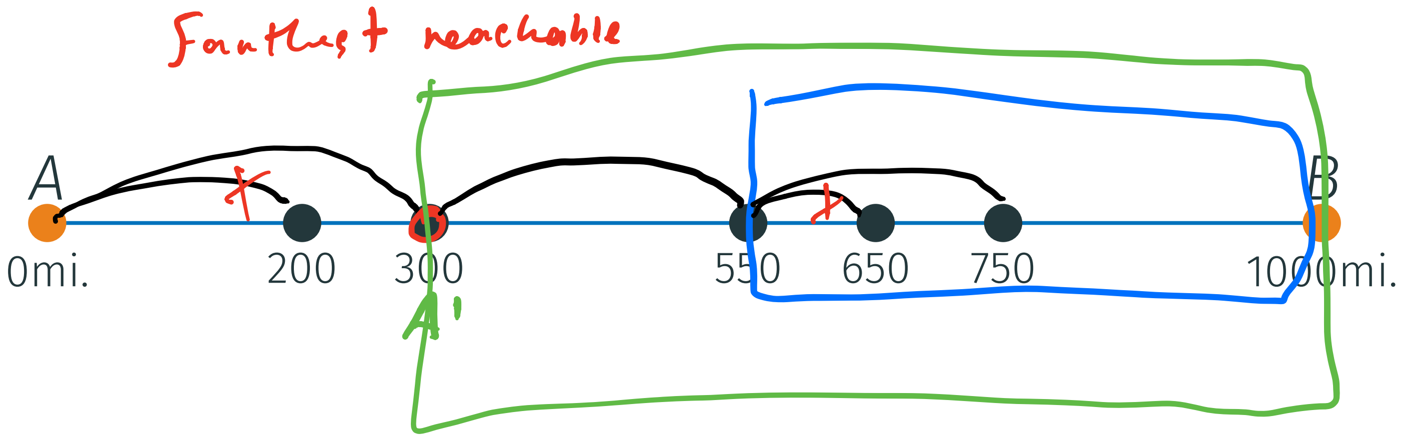


Break http://bit.ly/carfueling

Example

Distance with full tank 300 mi.

Minimize the number of stops at gas stations



Car Fueling. Solution

- “Greedy” algorithm

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- Runs in linear time $O(n)$, where n is the size of the input (# of gas stations)

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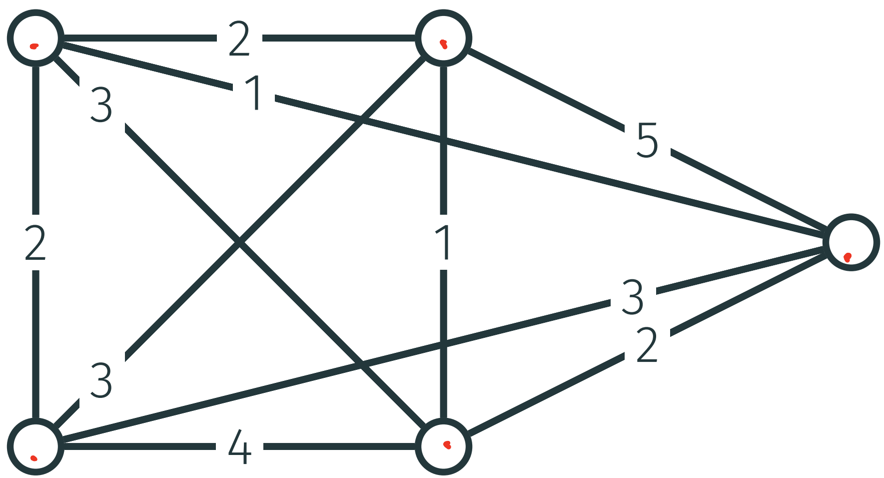
- "Greedy" algorithm
- Runs in linear time $O(n)$, where n is the size of the input (# of gas stations)
- Easy problem

Traveling Salesman Problem (TSP)

Traveling Salesman Problem

Given a complete weighted graph, find a cycle (or a path) of minimum total weight (length) visiting each node exactly once



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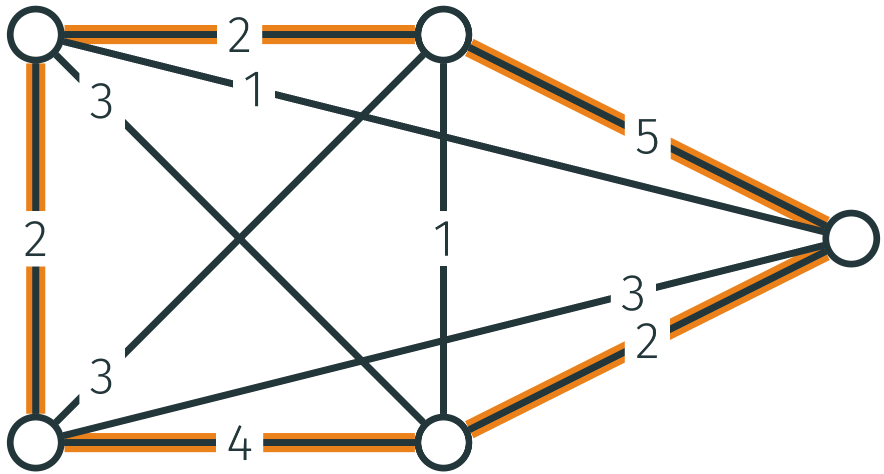

length: 15

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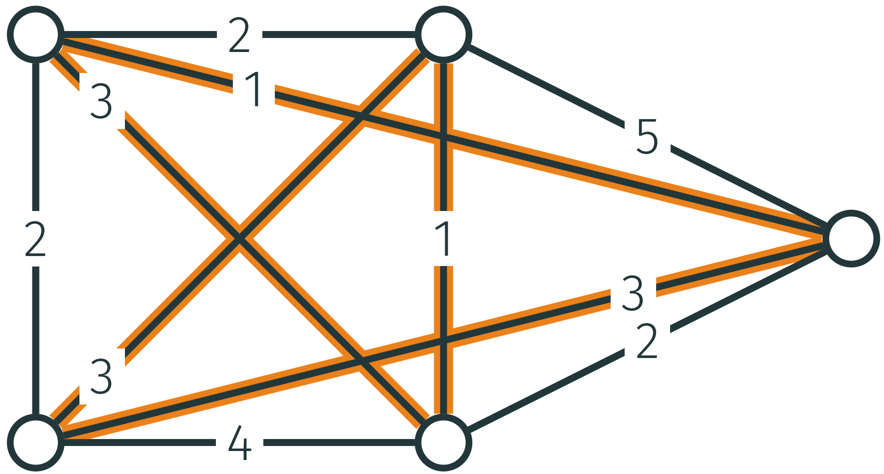

length: 11

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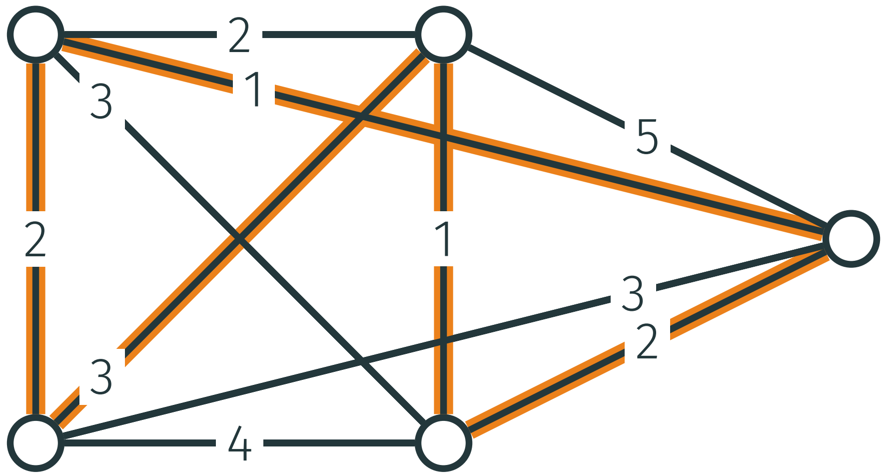

length: 9

Status

- Classical optimization problem with countless number of real life applications (we'll see soon)

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- No polynomial time algorithms known
- The best known algorithm runs in time 2^n

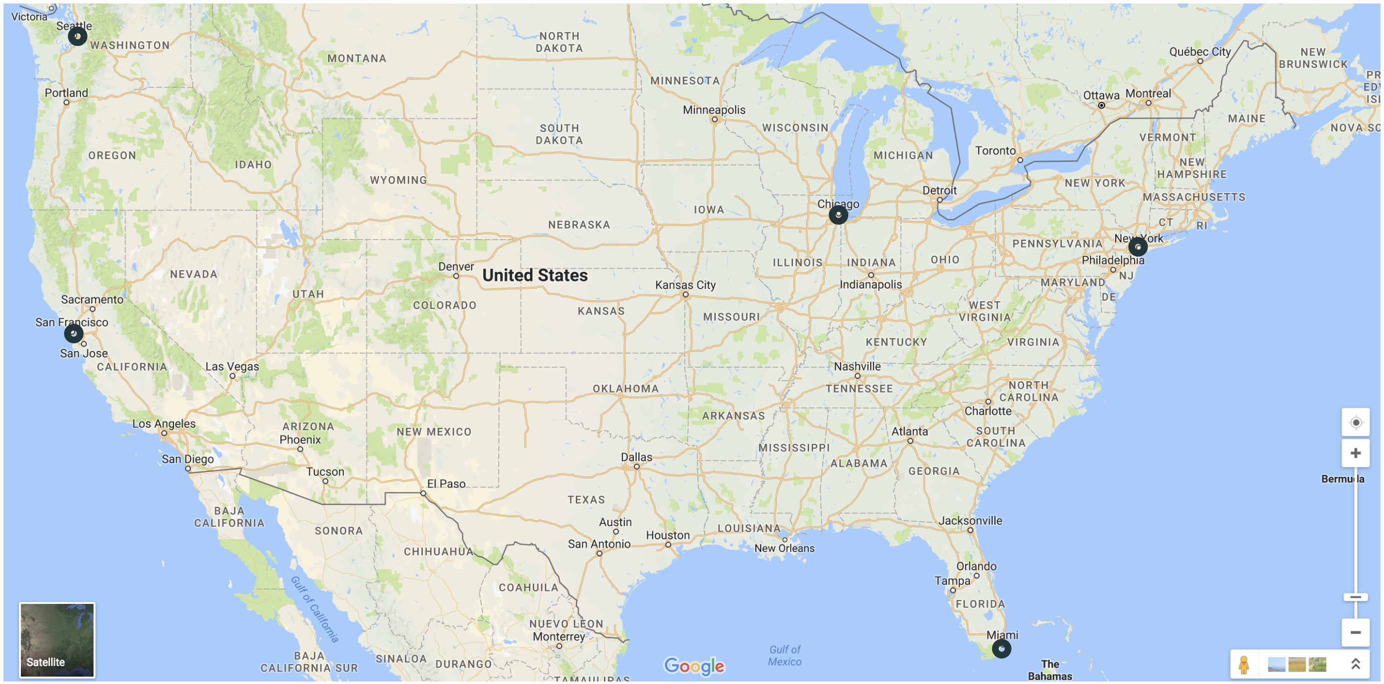
Delivering Goods

Need to visit several points. What is the optimal order of visiting them?

Traveling


Victoria
Seattle
WASHINGTON
MONTANA
NORTH DAKOTA
MINNESOTA
Québec City
NEW BRUNSWICK
Portland
Minneapolis
Ottawa
Montreal
MAINE
NOVA SC
OREGON
IDAHO
SOUTH DAKOTA
WISCONSIN
VERMONT
MICHIGAN
Toronto
NEW HAMPSHIRE
WYOMING
Detroit
NEW YORK
MASSACHUSETTS
IOWA
Chicago
CT
RI
NEBRASKA
PENNSYLVANIA
New York
NEVADA
Denver
United States
ILLINOIS
INDIANA
OHIO
Philadelphia
NJ
Kansas City
Indianapolis
MARYLAND
DE
UTAH
Sacramento
COLORADO
WEST VIRGINIA
KANSAS
MISSOURI
San Francisco
San Jose
KENTUCKY
VIRGINIA
CALIFORNIA
Las Vegas
Nashville
OKLAHOMA
TENNESSEE
NORTH CAROLINA
Charlotte
Los Angeles
ARIZONA
NEW MEXICO
ARKANSAS
Atlanta
SOUTH CAROLINA
Phoenix
San Diego
MISSISSIPPI
Dallas
ALABAMA
GEORGIA
Tucson
El Paso
TEXAS
Bermuda
BAJA CALIFORNIA
SONORA
Austin
LOUISIANA
Jacksonville
CHIHUAHUA
San Antonio
Houston
New Orleans
Orlando
Tampa
COAHUILA
FLORIDA
Gulf of California
NUEVO LEON
Monterrey
Miami
BAJA CALIFORNIA SUR
SINALOA
DURANGO
Google
Gulf of Mexico
The Bahamas
Satellite

Traveling


United States
Victoria
Seattle
WASHINGTON
Portland
OREGON
IDAHO
MONTANA
WYOMING
NORTH DAKOTA
SOUTH DAKOTA
NEBRASKA
MINNESOTA
Minneapolis
WISCONSIN
MICHIGAN
IOWA
Chicago
Detroit
Toronto
Ottawa
Montreal
Québec City
NEW BRUNSWICK
MAINE
NOVA SC
VERMONT
NEW HAMPSHIRE
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CT
RI
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NJ
MARYLAND
DE
OHIO
INDIANA
Indianapolis
ILLINOIS
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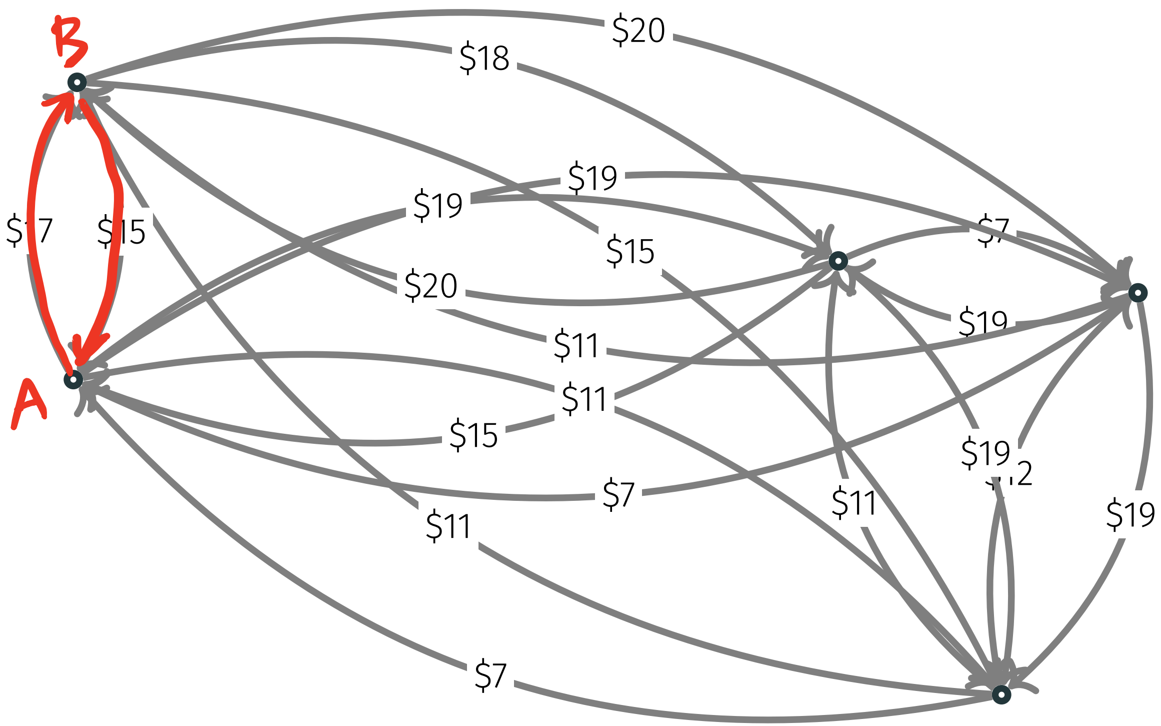
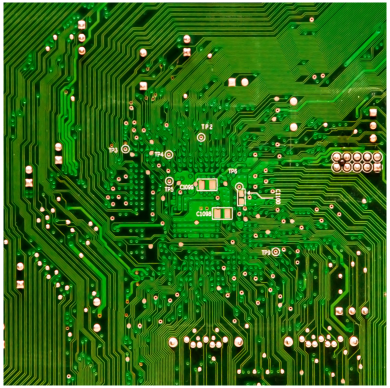


Drilling a Circuit Board



https://developers.google.com/optimization/routing/tsp/tsp

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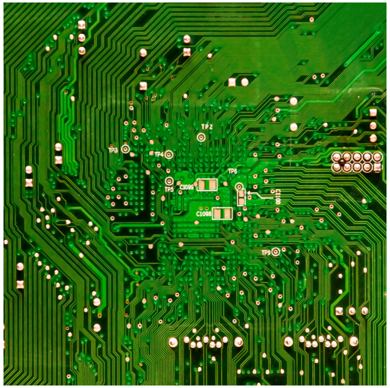

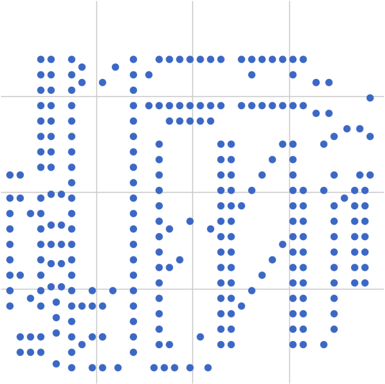

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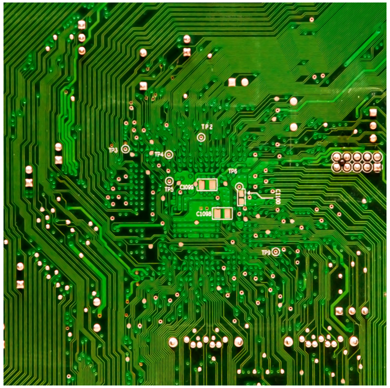

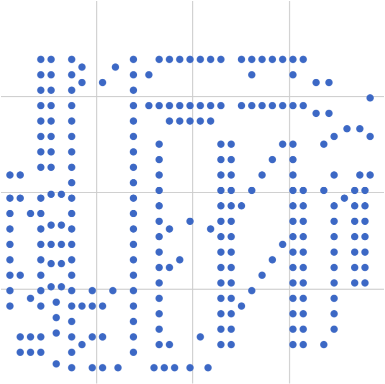

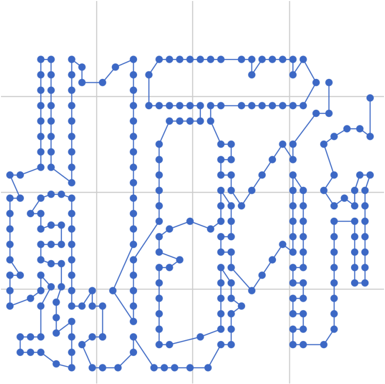

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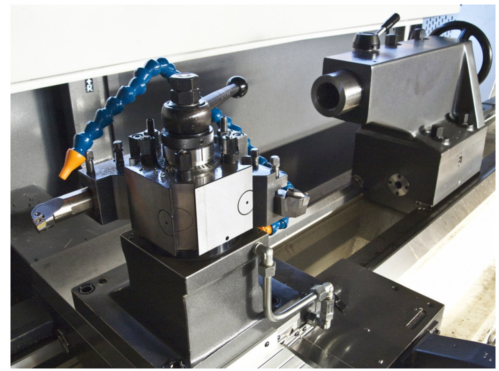
Processing Components

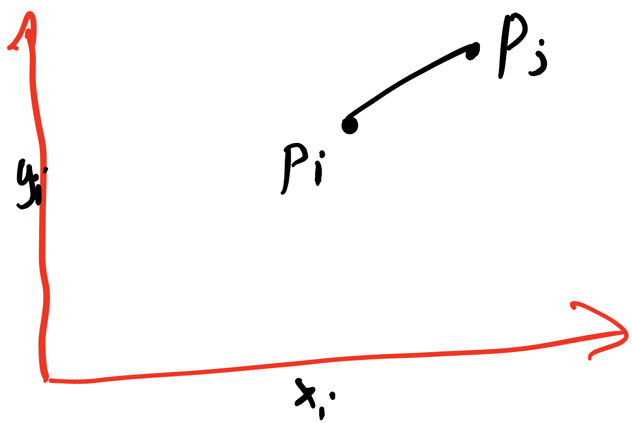
There are n mechanical components to
be processed on a complex machine. After processing the i-th component, it takes

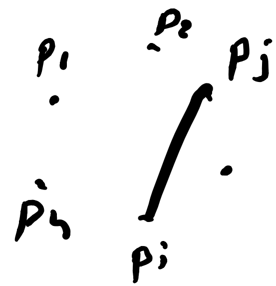


t_{ij} units of time to reconfigure the machine so that it is able to process the j-th component. What is the minimum processing cost?

Euclidean TSP

- Euclidean TSP: instead of a complete graph, the input consists of n points $p_1 = (x_1, y_1), \ldots, p_n = (x_n, y_n)$ on the plane





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- Weights are symmetric: $d(p_i, p_j) = d(p_j, p_i)$

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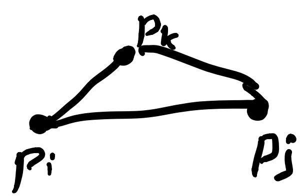
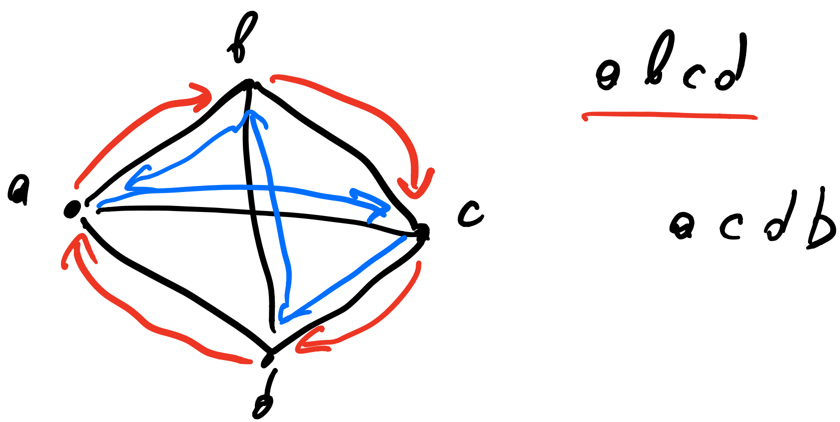
- Weights are symmetric: $d(p_i, p_j) = d(p_j, p_i)$
- Weights satisfy the triangle inequality: $d(p_i, p_j) \leq d(p_i, p_k) + d(p_k, p_j)$



Brute Force Search

- Finding the best permutation is easy: simply iterate through all of them and select the best one



Brute Force Search

- Finding the best permutation is easy: simply iterate through all of them and select the best one
- But the number of permutations of n objects is $n!$

$n!$: Growth Rate

n	$n!$
5	120
8	40320
10	3628800
13	6227020800
20	2432902008176640000
30	265252859812191058636308480000000

$\approx n!$ —later→ 2^n

Satisfiability Problem (SAT)

SAT

$x_1 \ldots x_n$ Boolean

x_i: True 1, False 0

$x_1 = 1 \quad x_2 = 1 \quad x_3 = 0$ doesn't satisfy formula

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3)$$

negation; clauses; OR; AND

each clause = OR vars or their negations

"1 / True"

Satisfy all clauses

SAT = does there exist assignment to $x_1, \ldots, x_n$ that satisfies all clauses?

SAT

YES: satisfiable

$x_1 = 1 \quad x_2 = 1 \quad x_3 = 1$

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3)$$

0 0 0 0 1 0 1

$$\rightarrow (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$$

NO: unsatisfiable

Case 1. $x_1 = 1 \Rightarrow x_3 = 1 \Rightarrow x_2 = 1$

Case 2. $x_1 = 0 \Rightarrow x_2 = 0 \Rightarrow x_3 = 0$

Applications of SAT

- Software Engineering
- Chip testing
- Circuit design
- Automatic theorem provers
- Image analysis
- ...

k-SAT

$$\phi(x_1, \ldots, x_n) = (x_1 \vee \neg x_2 \vee \ldots \vee x_k) \wedge$$
$$\ldots \wedge$$
$$(x_2 \vee \neg x_3 \vee \ldots \vee x_8)$$

k-SAT

$$\phi(x_1, \ldots, x_n) = (x_1 \vee \neg x_2 \vee \ldots \vee x_k) \quad \wedge$$
$$\ldots \quad \wedge$$
$$(x_2 \vee \neg x_3 \vee \ldots \vee x_8)$$

ϕ is satisfiable if

$$\exists x \in \{0,1\}^n \colon \phi(x) = 1\,.$$

Otherwise, ϕ is unsatisfiable

k-SAT

$$\begin{aligned}\phi(x_1,\ldots,x_n) = &(x_1 \vee \neg x_2 \vee \ldots \vee x_k) \quad \wedge \\ &\qquad \ldots \qquad\qquad \wedge \\ &(x_2 \vee \neg x_3 \vee \ldots \vee x_8)\end{aligned}$$

ϕ is satisfiable if

$$\exists x \in \{0,1\}^n \colon \phi(x) = 1\,.$$

Otherwise, ϕ is unsatisfiable

k-SAT is SAT where clause length $\leq k$

integer k

1-SAT
2-SAT
3-SAT

k-SAT. Examples

3-SAT

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3)$$

k-SAT. Examples

2-SAT

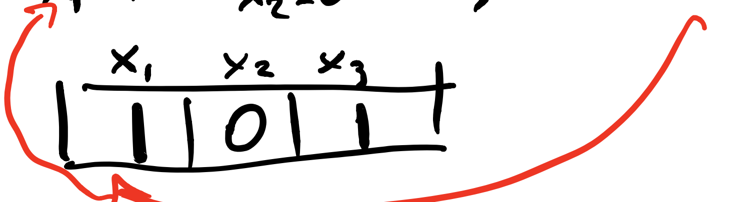
$(x_1 \lor \overline{x_2}) \land (\overline{x_3} \lor x_7)$

$$(x_1 \lor x_2 \lor x_3) \land (x_1 \lor \neg x_2) \land (\neg x_1 \lor x_3) \land (x_2 \lor \neg x_3)$$

1-SAT

$\rightarrow \quad (x_1) \land (\neg x_2) \land (x_3) \land (\neg x_1)$

$x_1 = 1 \qquad x_2 = 0 \qquad x_3 = 1 \qquad x_1 = 0$



Queen of NP-complete problems

- Cook-Levin Theorem [Coo71, Lev73]: SAT can model non-deterministic Turing machine:

  SAT is NP-complete

Queen of NP-complete problems

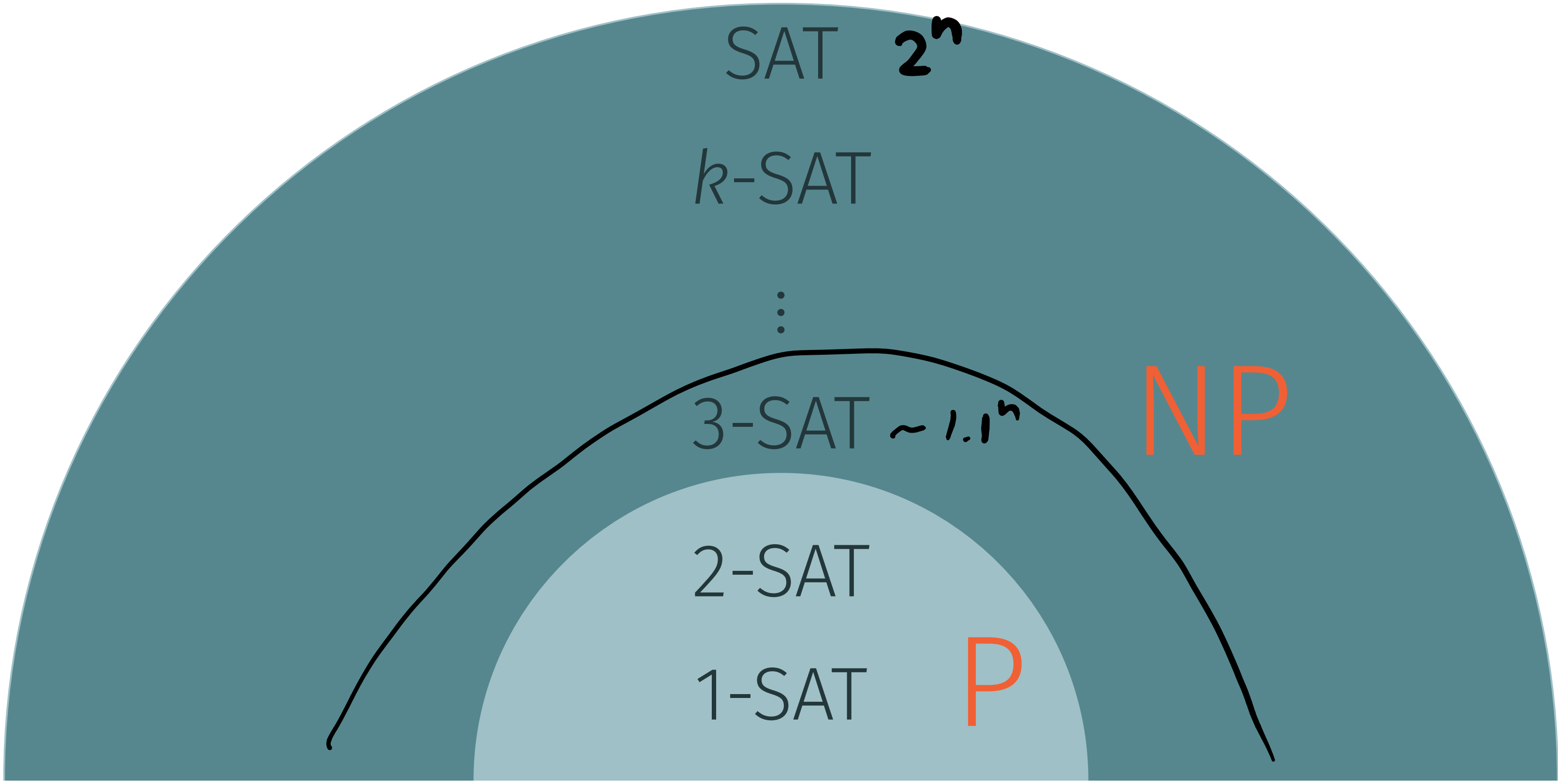
- Cook-Levin Theorem [Coo71, Lev73]: SAT can model non-deterministic Turing machine: SAT is NP-complete
- 3-SAT is NP-complete

Queen of NP-complete problems

- Cook-Levin Theorem [Coo71, Lev73]: SAT can model non-deterministic Turing machine: SAT is NP-complete
- 3-SAT is NP-complete
- 2-SAT is in P
  1-SAT is in P

Complexity of SAT

Complexity of SAT



The SAT game

http://bit.ly/sat-game