Gems of TCS

Integer Linear Programming

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March 4, 2021

Avoiding Scurvy

- Orange costs \$1,
  grapefruit costs \$1;
  we have budget of \$2/day
- Orange weighs 100gm,
  grapefruit weighs 200gm,
  we can carry 300gm
- Orange has 100gm of vitamin C,
  grapefruit has 150gm of vitamin C,
  maximize daily vitamin C intake.

Avoiding Scurvy. Plot

$\max 2x + 3y$

$$
\begin{aligned}
x + y &\leq 2 \\
x + 2y &\leq 3 \\
x &\geq 0 \\
y &\geq 0
\end{aligned}
$$

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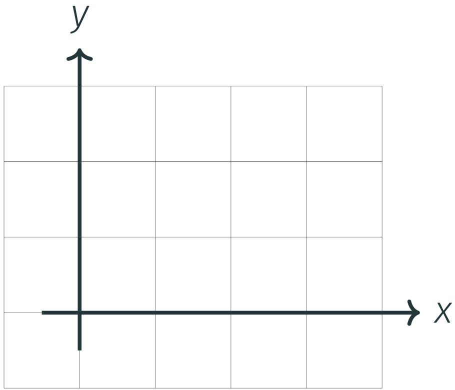

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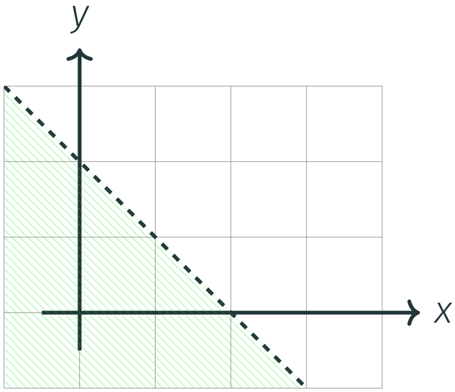

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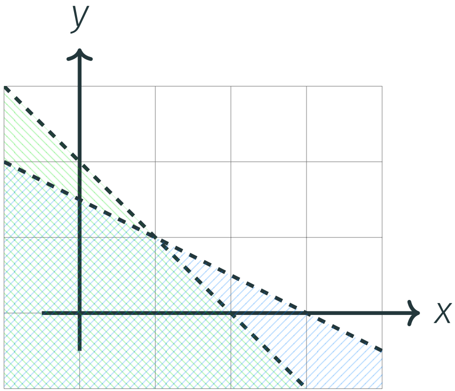

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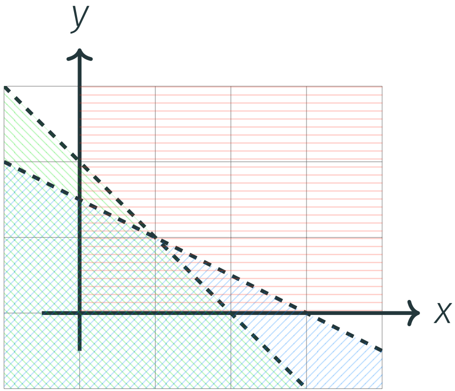

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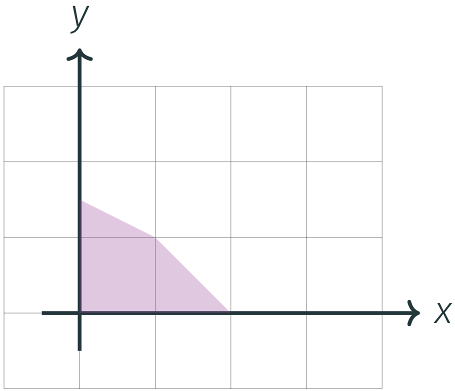

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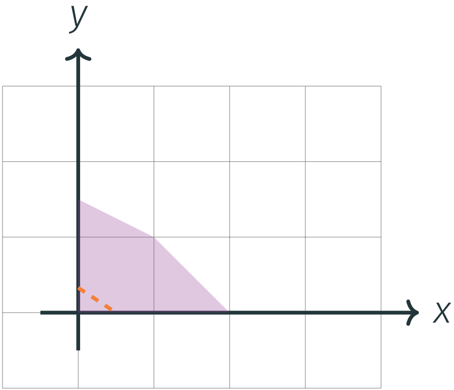

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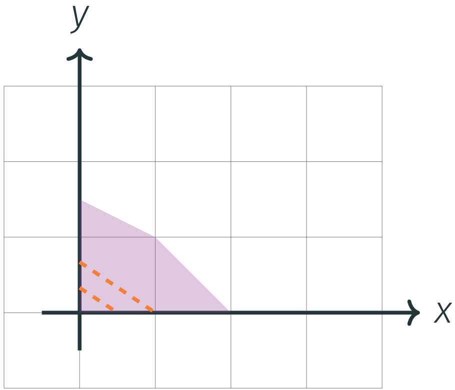

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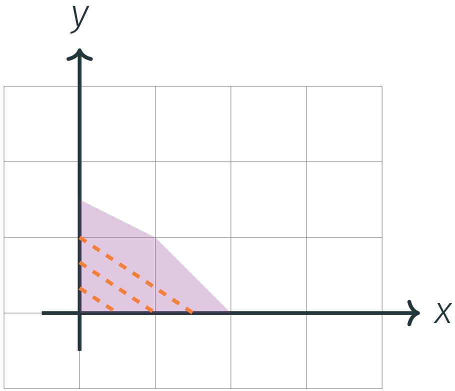

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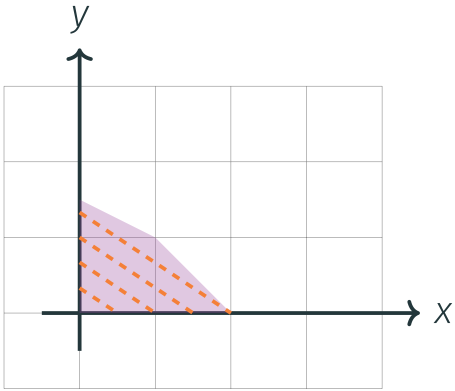

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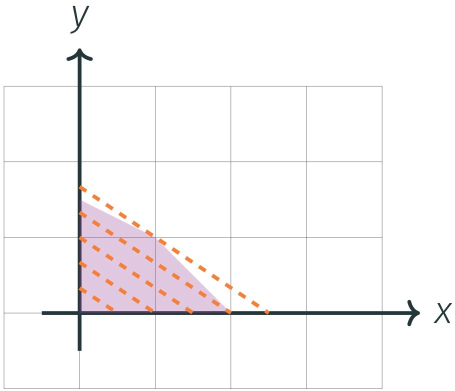

Avoiding Scurvy II

$$\max 2x + 3y$$

$$\begin{aligned} x + y &\leq 2 \\ x + 2y &\leq 2.5 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

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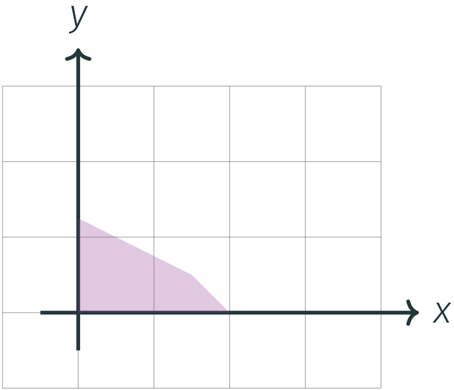

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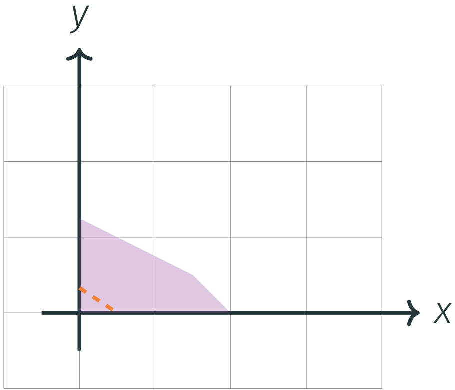

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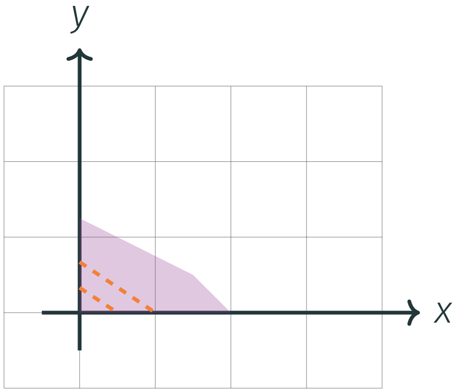

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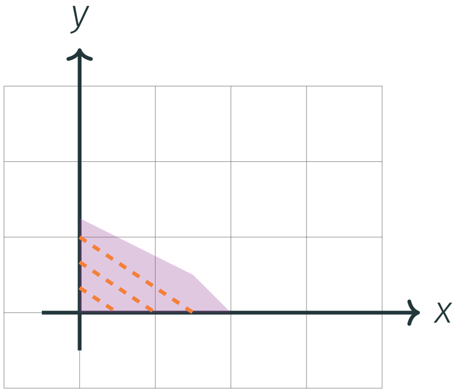

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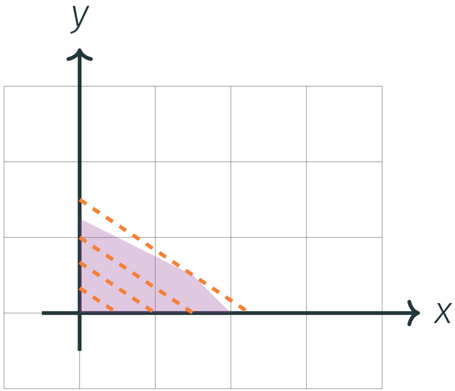

Linear programming

Input: A set of linear inequalities $Ax \leq b$.

Output: Real solution that optimizes the objective function.

Integer linear programming

Input: A set of linear inequalities $\mathbf{Ax} \leq \mathbf{b}$.

Output: Integer solution that optimizes the objective function.

Example

$$
\begin{aligned}
x_1 &\geq 0.5 \\
-x_1 + 8x_2 &\geq 0 \\
-x_1 - 8x_2 &\geq -8
\end{aligned}
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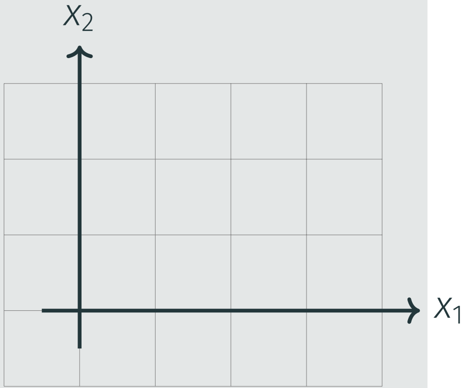

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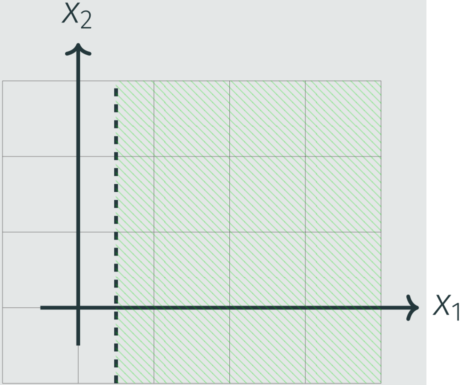

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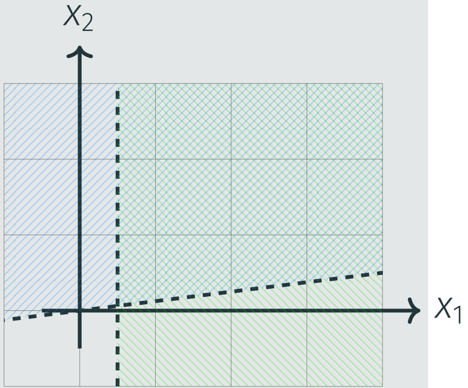

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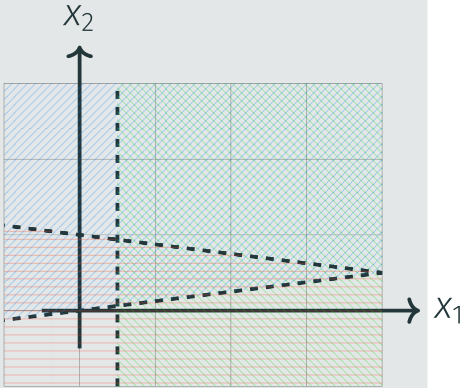

LP

Find a real solution of a system of linear inequalities

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Can be solved efficiently (Lecture 9)

LP	ILP
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LP	ILP
Find a real solution of a system of linear inequalities	Find an integer solution of a system of linear inequalities
Can be solved efficiently (Lecture 9)	No polynomial algorithm known!

Algorithm for ILP

$$\max 2x + y$$

$$\begin{aligned} 4x + y &\le 33 \\ 3x + 4y &\le 29 \\ x &\ge 0 \\ y &\ge 0 \\ x, y &\in \mathbb{Z} \end{aligned}$$

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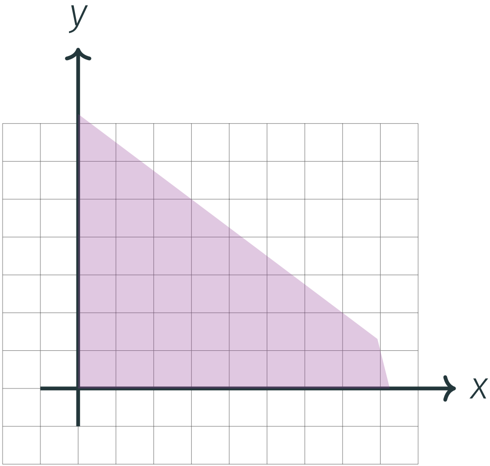


Linear Programming Solver

BETA

Linear Programming ✕

(max: 9 variables)

Optimize:	Max	
Objective Function:	2x+y	
Subject to:	4x+y<=33,	3x+4y<=29,
	x>=0,	
and:	y>=0	
More constraints(optional):		
More constraints(optional):		

Solve (multiple constr. in a box are allowed)

www.ordsworks.com *(constraints separator: ",")

Global maximum:

Exact form | More digits

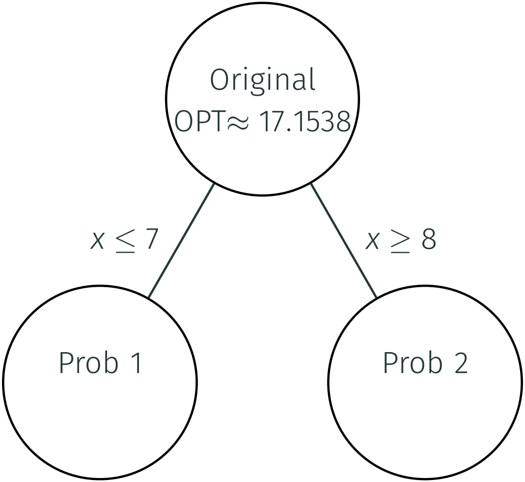
$\max\{2x + y \mid 4x + y \le 33 \land 3x + 4y \le 29 \land x \ge 0 \land y \ge 0\} \approx 17.1538$ at $(x, y) \approx (7.92308, 1.30769)$

Branching on x

Original

OPT$\approx$ 17.1538

Branching on x



Branching

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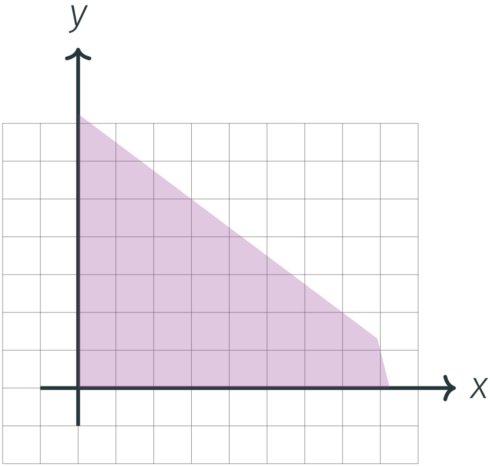

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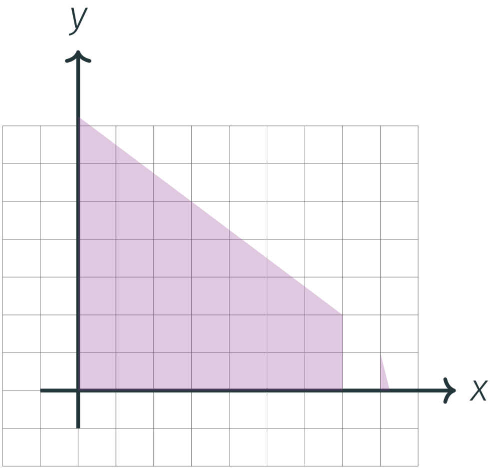


Linear Programming Solver

BETA

Linear Programming

(max: 9 variables)

Optimize: Max

Objective Function: 2x+y

Subject to: 4x+y<=33, 3x+4y<=29,

x>=0,

and: y>=0,

More constraints(optional): x<=7

More constraints(optional):

Solve (multiple constr. in a box are allowed)

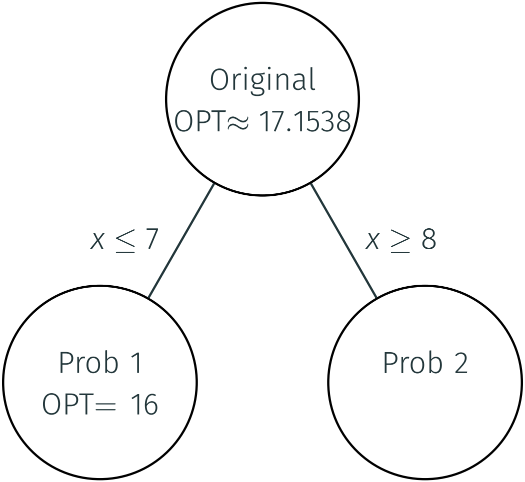
www.ordsworks.com *(constraints separator: ",")

Global maximum:

$$\max\{2x + y \mid 4x + y \leq 33 \land 3x + 4y \leq 29 \land x \geq 0 \land y \geq 0 \land x \leq 7\} = 16$$

$$\text{at } (x, y) = (7, 2)$$

Branching on x



Linear Programming Solver

BETA

Linear Programming

(max: 9 variables)

Optimize:	Max	
Objective Function:	2x+y	
Subject to:	4x+y<=33,	3x+4y<=29,
	x>=0,	
and:	y>=0,	
More constraints(optional):	x>=8	
More constraints(optional):		

Solve

(multiple constr. in a box are allowed)

www.ordsworks.com

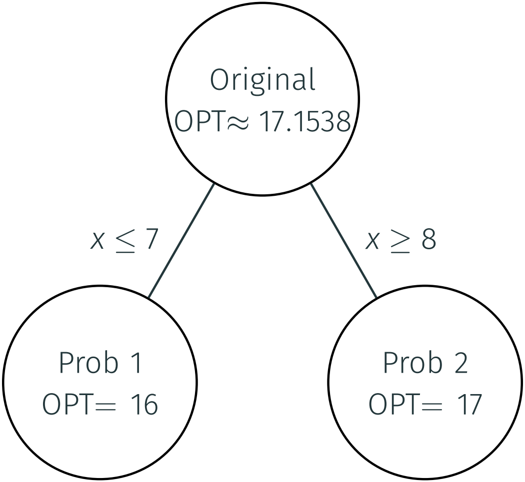
*(constraints separator: ",")

Global maximum:

$$\max\{2x + y \mid 4x + y \le 33 \land 3x + 4y \le 29 \land x \ge 0 \land y \ge 0 \land x \ge 8\} = 17$$

at $(x, y) = (8, 1)$

Branching on x



Heuristic Algorithms for ILP

Applications

Applications

- Scheduling
- Planning
- Networks
- ...

Vertex Covers

- A Vertex Cover of a graph G is a set of vertices C such that every edge of G is connected to some vertex in C.

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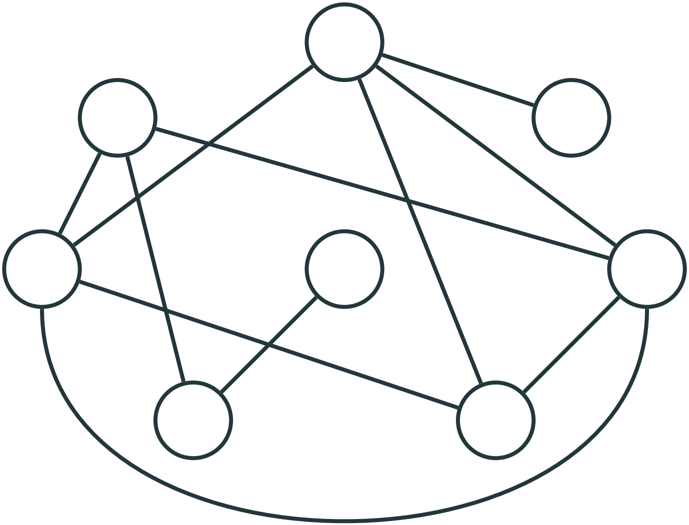
- A Minimum Vertex Cover is a vertex cover of the smallest size.

Vertex Covers: Examples



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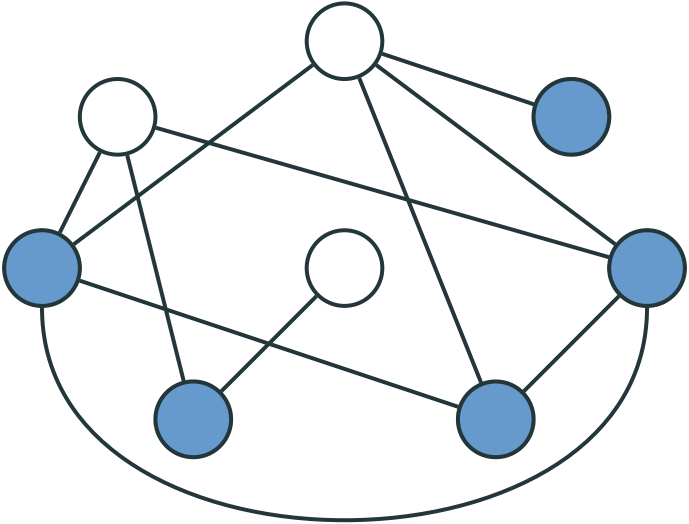

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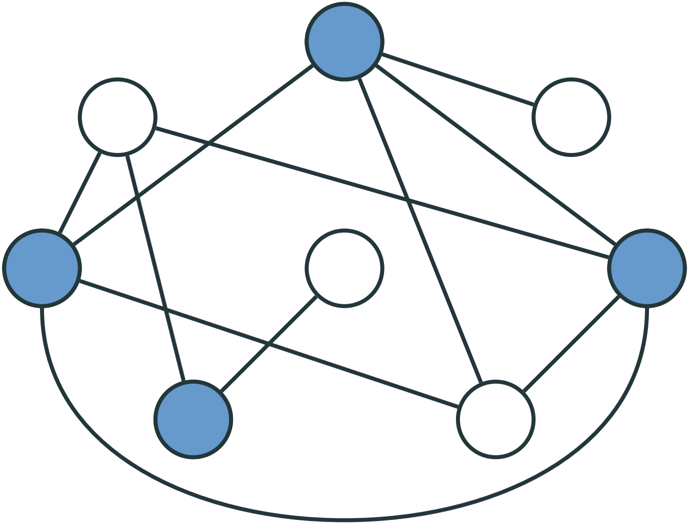

Vertex Cover as ILP

- Introduce binary variable for every vertex: $x_1, \ldots, x_n$:
    - $x_i = 1$ iff x_i belongs to Vertex Cover

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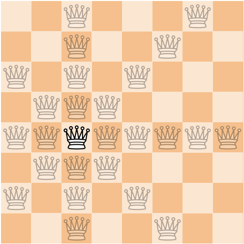
- $\forall i \in \{1, \ldots, n\},\ 0 \leq x_i \leq 1, x_i \in \mathbb{Z}$
- $\min \sum_i x_i$
- For every edge (u, v) in th graph: $x_u + x_v \geq 1$

Implementation

```python
import networkx as nx
from mip import *

g = nx.Graph()
g.add_edges_from([(1, 2), (1, 3), (1, 5), (1, 6), (2, 5), (2, 0),
                  (3, 4), (3, 5), (3, 6), (5, 6), (7, 0)])

m = Model()
n = g.number_of_nodes()
x = [m.add_var(var_type=BINARY) for i in range(n)]
for u, v in g.edges():
    m += x[u]+x[v] >= 1
m.objective = minimize(xsum(x[i] for i in range(n)))
m.optimize()

selected = [i for i in range(n) if x[i].x >= 0.99]
print("selected items: {}".format(selected))
```

N Queens

Is it possible to place n queens on an $n \times n$ board such that no two of them attack each other?



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$$\sum_{j=1}^{n} x_{ij} = 1\,.$$

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$$\sum_{j=1}^{n} x_{ij} = 1 .$$
- For $0 \leq j < n$, jth column contains $= 1$ queen:
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- For $0 \leq j < n$, jth column contains $= 1$ queen:
$$\sum_{j=1}^{n} x_{ij} = 1\,.$$
- Each diagonal contains ≤ 1 queen:
$$\sum_{i=1}^{n} \sum_{j=1\colon\, i-j=k}^{n} x_{ij} \leq 1\,;\;\; \sum_{i=1}^{n} \sum_{j=1\colon\, i+j=k}^{n} x_{ij} \leq 1$$