Gems of TCS

Gödel's Incompleteness

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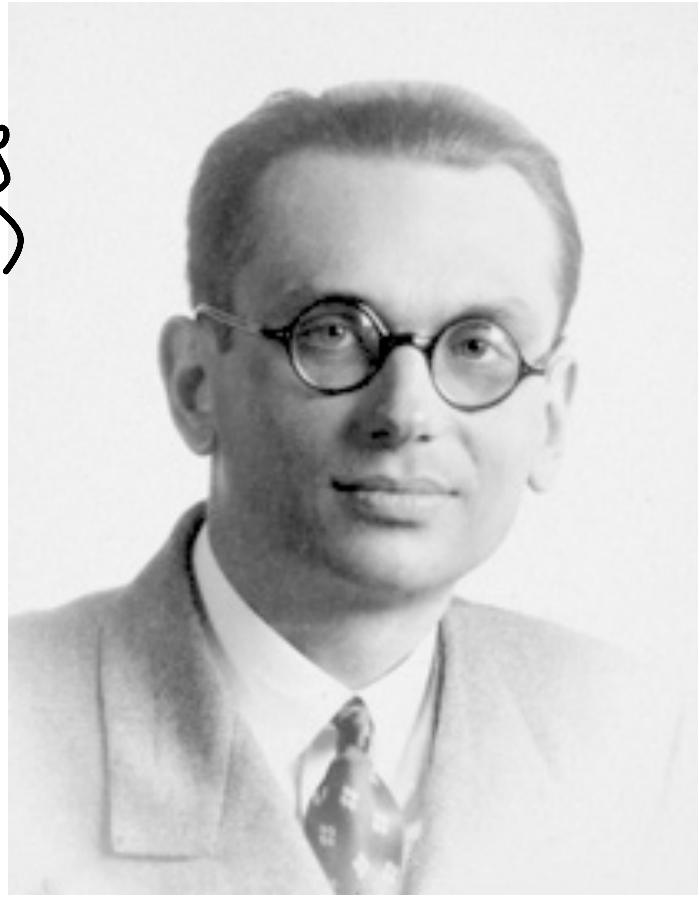
Godel's Incompleteness Theorem

1931 Gödel

1936 Turing

(Lecture 13)



$x = y$ &

$y = z$

$\Downarrow$

$x = z$

Axiomatization of Math

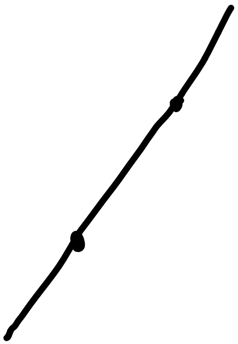
- Find a set of simple and obvious axioms

Axiomatization of Math

- Find a set of simple and obvious axioms
- Any proof could be (in principle) traced back to this set of axioms

Euclid's Axioms

- For any pair of distinct points, there is exactly one line connecting them



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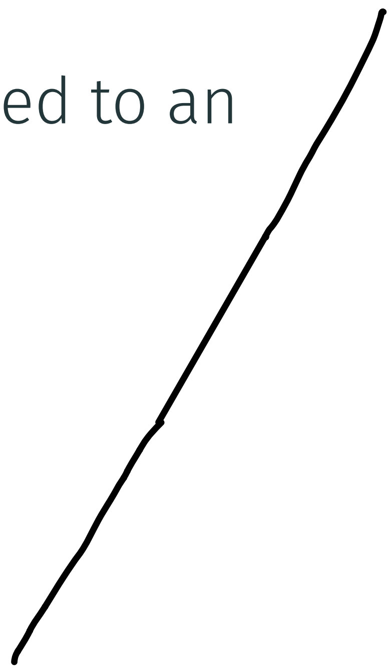

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- Any line segment can be extended to an infinite line
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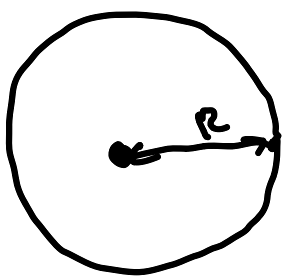

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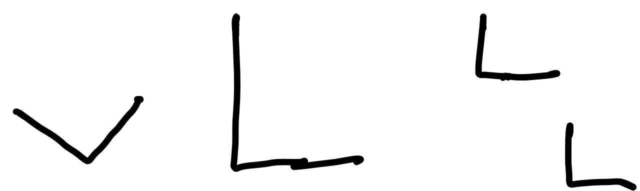

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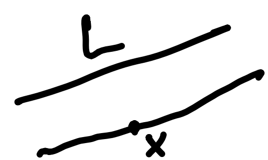

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Peano Arithmetic

- 0 is a natural number

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- $\forall x,\ x = x$
- If $x = y$, then $y = x$
- If $x = y$ and $y = z$, then $x = z$
- $\dots$

Peano Arithmetic

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- If $x = y$ and $y = z$, then $x = z$
- ...
- $\forall x, y,\ x = y$ iff $\mathsf{Next}(x) = \mathsf{Next}(y)$
- If x is a natural number, then $\mathsf{Next}(x)$ is a natural number
- There is no natural x s.t. $Next(x) = 0$
- ...

$Next(x) = x + 1$

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- Induction

Naive Set Theory

Cantor, Dedekind, ...

- Set
- Membership in a Set
- Empty Set $\emptyset$
- Equality $A = B$ iff they contain same elements

Russell's paradox

S = set of all sets that don't contain themselves as an element

Case 1: S contains itself as an element

⇒ S cannot contain S as an element

Case 2: S doesn't contain itself as an element

⇒ S must contain S as an element.

T = "S contains itself"

$\neg T$ = "S doesn't contain itself"

We can prove T and $\neg T$.

T is true $\Rightarrow$

$\cancel{T}$ OR "I'm the Pope" is true

$\neg T$ is true

"I'm the Pope"

Russell's paradox

$S =$ contains all sets that don't contain themselves as an element

The barber is the ”one who shaves all those, and those only, who do not shave themselves”. The question is, does the barber shave himself?

Principia Mathematica

Bertrand & Whitehead

objects can't contain objects of the same type

Took 379 pages to prove $1+1=2$

Proof assitants (Coq, Isabella, ...)

1. Verify formal proofs
2. Help us to prove theorems

ZFC

Gold Standard

Zermelo & Fraenkel

Set Theory

$\Downarrow$

Integers $\Rightarrow$ Tuples $\quad (x, y) = 2^x \cdot 3^y$

$\Downarrow$

Reals

Tuples $\Rightarrow$ Graphs

Godel's Incompleteness Theorem

Any attempt to axiomatize all of mathematics is guaranteed to fail

Halting Problem

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  - $\text{HALT}(A, x) = 1$ if A halts on input x
  - $\text{HALT}(A, x) = 0$ if A enters infinite loop on input x
- HALT is undecidable (Lecture 13)

Gödel's Incompleteness Theorem

Any formal system is unsound OR incomplete.

sound ≡ proves only true statements

unsound ≡ proves some false statements

complete ≡ for every statement T, it proves T OR $\neg T$

incomplete ≡ there is a statement T s.t. our system cannot prove T nor $\neg T$.

EQ: Any sound system must be incomplete.

Believe ZFC is sound.

Alg. For Halting Problem (A, x)

Brute Force all strings

For each string s

- check s is a proof (in ZFC) that $A(x)$ halts: Output 1 Exit
- check s is a proof (in ZFC) that $A(x)$ runs forever: Output 0 Exit

---

Case I: we output 0 instead of 1 or output 1 instead of 0

$A(x)$ doesn't halt, but we have a proof that $A(x)$ halts

$\Rightarrow$ formal system is unsound

Case II: sometimes (for some A and X) blue algorithm runs forever

$\Rightarrow$ formal system is incomplete

Two examples of statements that cannot be proved or disproved in ZF (ZFC):

- Axiom of Choice
- Continuum Hypothesis

$$|\mathbb{N}| < \; ? \; < |\mathbb{R}|$$

Gödel's incompleteness theorem

ZFC

$T =$ "ZFC doesn't prove $\underbrace{\text{this statement}}_{(T)}$"