Gems of TCS

Circuit Complexity II

Sasha Golovnev

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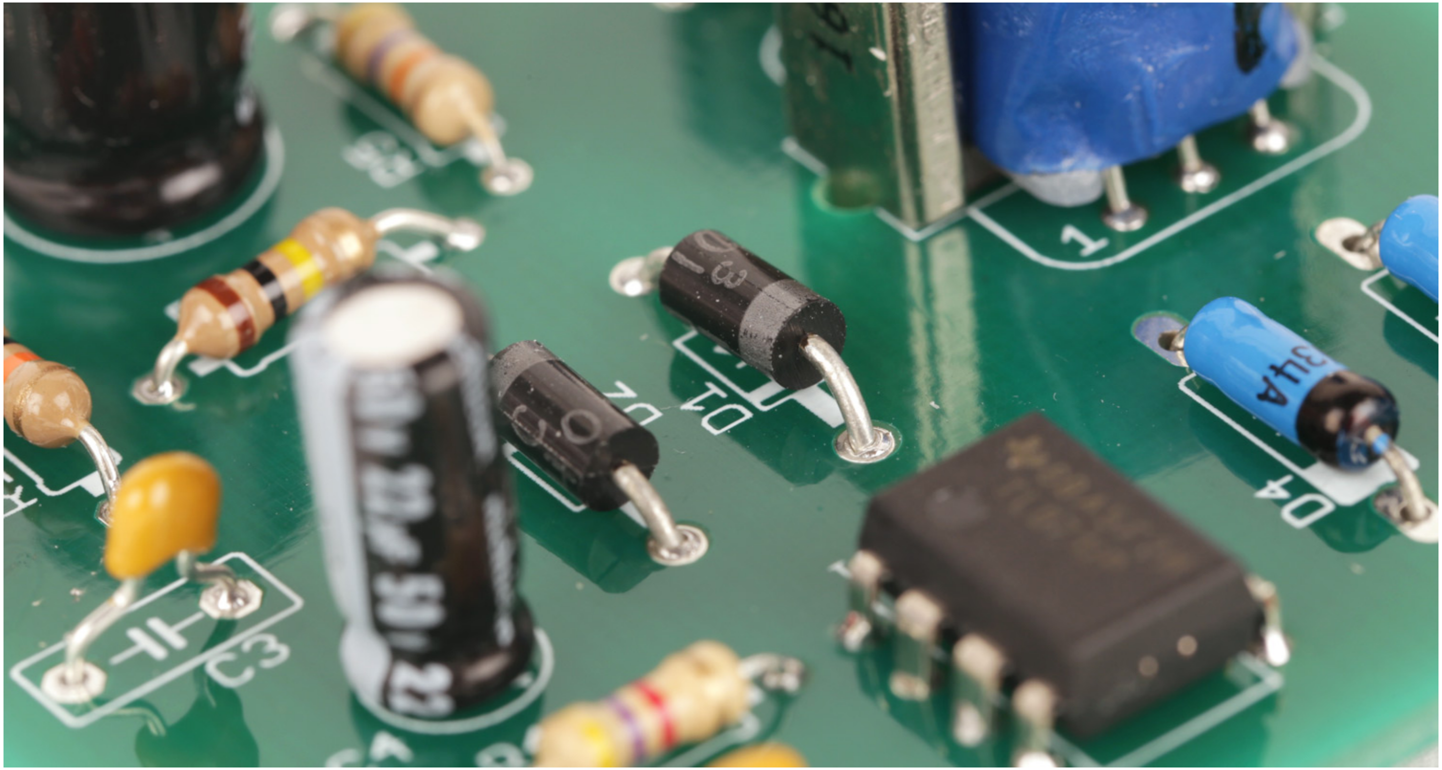
C3
D1
D2
D4

The main open problem in Computer Science

Is **P** equal to **NP**?

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- If P=NP, then all search problems can be solved in polynomial time.

The main open problem in Computer Science

Is **P** equal to **NP**?

- If **P**=**NP**, then all search problems can be solved in polynomial time.
- If **P**≠**NP**, then there exist search problems that cannot be solved in polynomial time.

Boolean Circuits

$$f \colon \{0,1\}^n \to \{0,1\}$$

Straight-line program

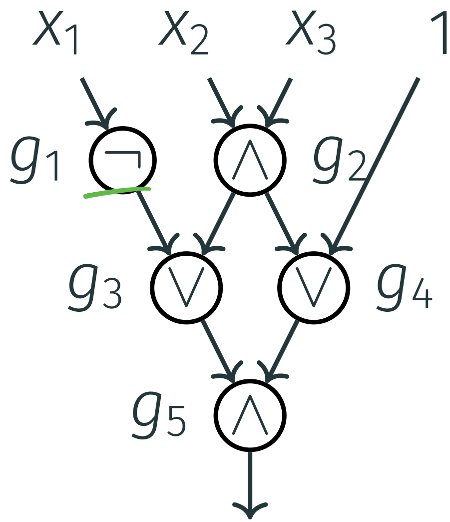
$$
\begin{aligned}
g_1 &= \neg x_1 \\
g_2 &= x_2 \wedge x_3 \\
g_3 &= g_1 \vee g_2 \\
g_4 &= g_2 \vee 1 \\
g_5 &= g_3 \wedge g_4
\end{aligned}
$$

Boolean Circuits

$$f\colon \{0,1\}^n \to \{0,1\}$$

acyclic graph

$$\begin{aligned} g_1 &= \neg x_1 \\ g_2 &= x_2 \wedge x_3 \\ g_3 &= g_1 \vee g_2 \\ g_4 &= g_2 \vee 1 \\ g_5 &= g_3 \wedge g_4 \end{aligned}$$



Exponential Bounds

$f : \{0, 1\}^n \rightarrow \{0, 1\}$

Lower Bound [Sha1949]

Almost all functions of n variables have circuit size

$$\geq 2^n / n$$

Exponential Bounds

Lower Bound [Sha1949]

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Almost all functions of n variables have circuit size

$$\boxed{\geq 2^n/n}$$

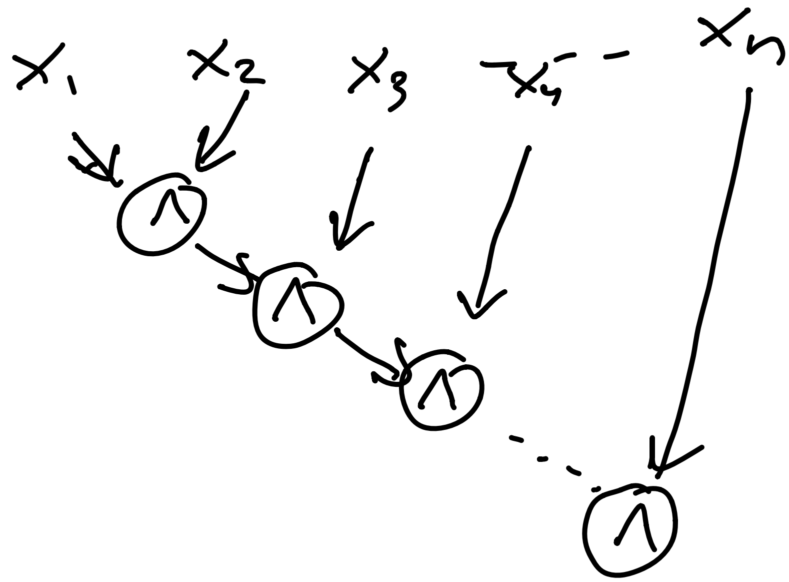
Upper Bound [Lup1958]

$f: \{0,1\}^n \to \{0,1\}$

Any function can be computed by a circuit of size

$$\leq 2^n/n$$

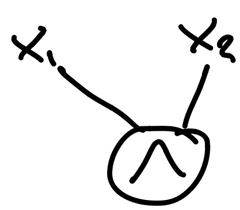
$$f(x_1, \dots, x_n) = \underline{x_1 \wedge x_2 \wedge x_3 \wedge \dots \wedge x_n}$$

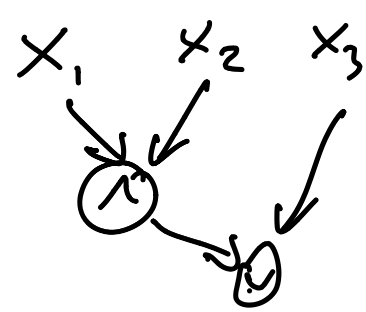


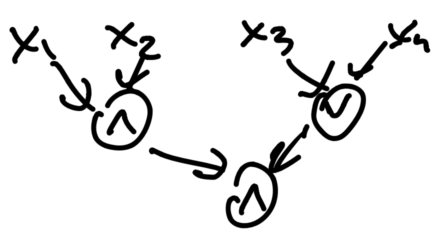
$n-1$ gates

$$\left\{\begin{array}{l} \text{size}(AND_n) \le n-1 \\ \text{size}(AND_n) \ge n-1 \end{array}\right.$$

in-deg(gate) $= 2$







circuit of size S can depend on $\le S+1$ inputs $\Rightarrow \text{size}(AND_n) \ge n-1$

Symmetric Funs

$x_1, \ldots, x_n \in \{0,1\}$

$AND_n(x_1, \ldots, x_n) = 1$ iff $x_1 = x_2 = \ldots = x_n = 1$

$\Updownarrow$

$\boxed{\sum_{i=1}^{n} x_i = n}$

$OR_n(x_1, \ldots, x_n) = 0$ iff $x_1 = x_2 = \ldots = x_n = 0$

$\Updownarrow$

$\boxed{\sum_{i=1}^{n} x_i = 0}$

$OR_n(x_1, \ldots, x_n) = 1 \iff \sum_{i=1}^{n} x_i > 0$

$XOR_n(x_1, \ldots, x_n) = 1$ iff $\boxed{\sum_{i=1}^{n} x_i}$ is odd

Symmetric Functions

$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric if it depends only on the number of ones in the input, and not on positions of these ones.

$f(0010) = f(1000) = f(0001) = f(0100)$

Symmetric Functions

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- $\text{And}(x) = 1$ iff $x_1 = \ldots = x_n = 1$

Σx_i

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XOR

- $\bigoplus(x) = 1$ iff $x_1 + \ldots + x_n \equiv 1 \mod 2$

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- $k \in \mathbb{N}$ $\mathrm{Mod}_k(x) = 1$ iff $\underline{x_1 + \ldots + x_n} \equiv 0 \mod k$

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- $\mathrm{Maj}(x) = 1$ iff $x_1 + \ldots + x_n \geq n/2$

Symmetric Functions

$f \colon \{0,1\}^n \to \{0,1\}$ is $\boxed{\text{symmetric}}$ if it depends only on the number of ones in the input, and not on positions of these ones.

- $\text{And}(x) = 1$ iff $x_1 = \ldots = x_n = 1$
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- $\text{Mod}_k(x) = 1$ iff $x_1 + \ldots + x_n \equiv 0 \mod k$
- $\text{Maj}(x) = 1$ iff $x_1 + \ldots + x_n \geq n/2$
- $k \in \mathbb{N}$ $\text{Th}_k(x) = 1$ iff $x_1 + \ldots + x_n \geq k$

Threshold$_k$ | $\text{Maj} = \text{Th}_{n/2}$ | $\text{AND} = \text{Th}_n$ | $\text{OR} = \text{Th}_1$

Symmetric Functions. Equiv Def

$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric iff

$$f = g(x_1 + \ldots + x_n)$$

for some $g\colon \{0,\ldots,n\} \to \{0,1\}$.

$g : \{0,1\}^{\log_2(n+1)} \to \{0,1\}$

$f = AND_n$ $\quad f = g(x_1 + \ldots + x_n)$

$\begin{cases} g(n) = 1 \\ g(i) = 0 \end{cases} \quad \forall i \in \{0, 1, \ldots, n-1\}$

$f = OR_n$ $\quad f = g(x_1 + \ldots + x_n)$

$\begin{cases} g(0) = 0 \\ g(i) = 1 \end{cases} \quad \forall i \in \{1, 2, \ldots, n-1\}$

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$f\colon \{0,1\}^n \to \{0,1\}$ is symmetric iff

$$f = \boxed{h}(\underline{\mathrm{Sum}_n(x_1,\ldots,x_n)})$$

$h(b_1, b_2, \ldots, b_{\log_2 n}) =$

for some $h\colon \{0,1\}^{\log n} \to \{0,1\}$, where $= g(b_1 + 2b_2 + 4b_3 + \ldots)$

$\mathrm{Sum}_n\colon \{0,1\}^n \to \{0,1\}^{\log n}$.

Complexity of Sum

$\mathrm{Sum}_3 : \{0,1\}^3 \to \{0,1\}^2$

$\mathrm{Sum}_3(x_1, x_2, x_3) \in \{0,1\}^2$

Complexity of Sum

$$\mathsf{Sum}_3(x_1, x_2, x_3) \in \{0, 1\}^2$$

$$\mathsf{Sum}_3(x_1, x_2, x_3) = (\mathsf{carry}, \mathsf{parity})$$

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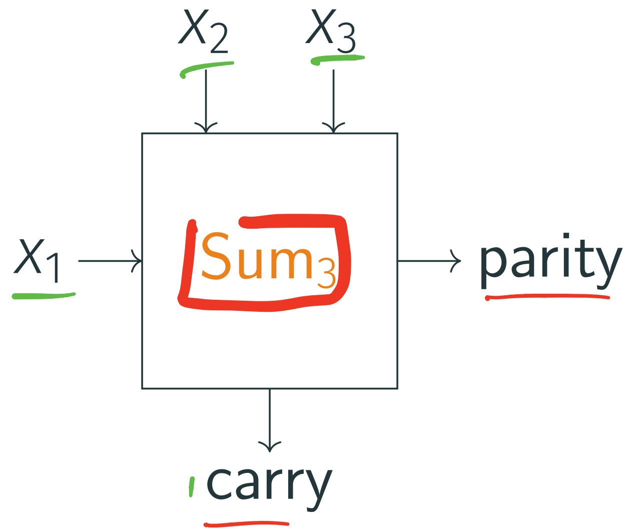

$\mathrm{size}(\mathrm{Sum}_3) \leq O(1)$

Sum_n

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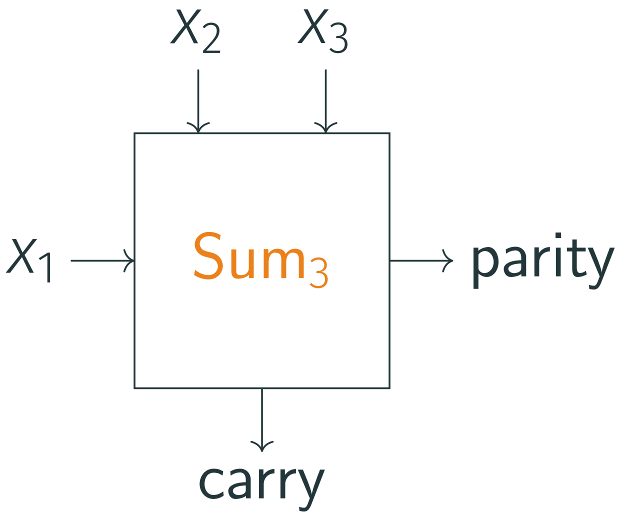

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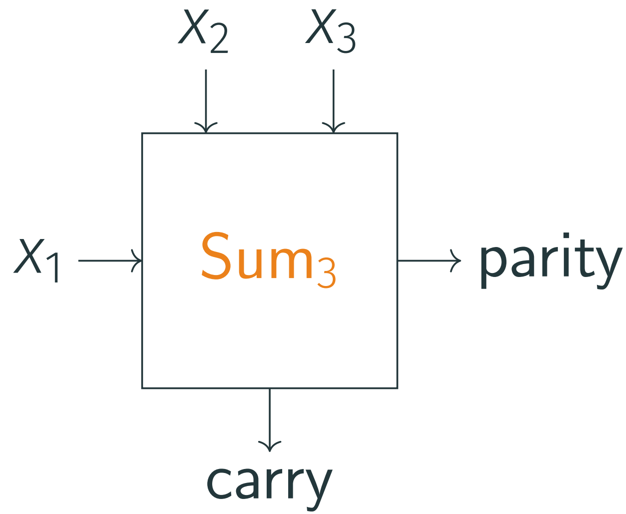

Sum_5?

Complexity of Sum

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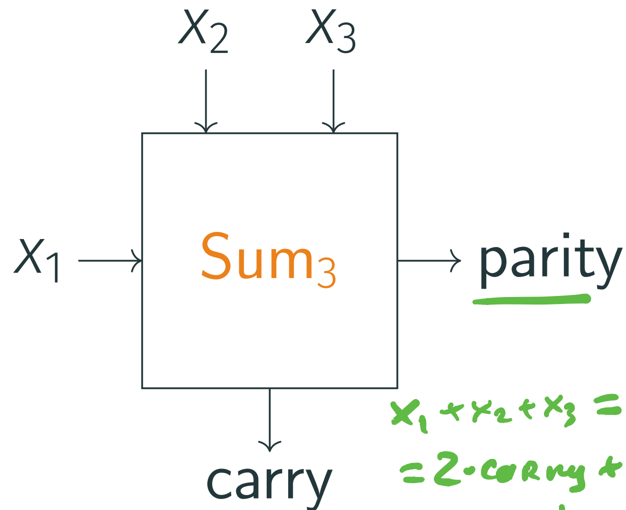
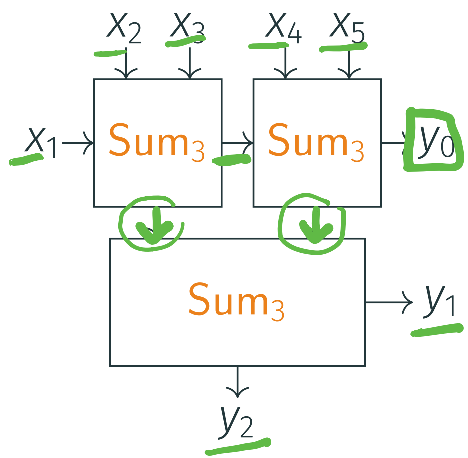


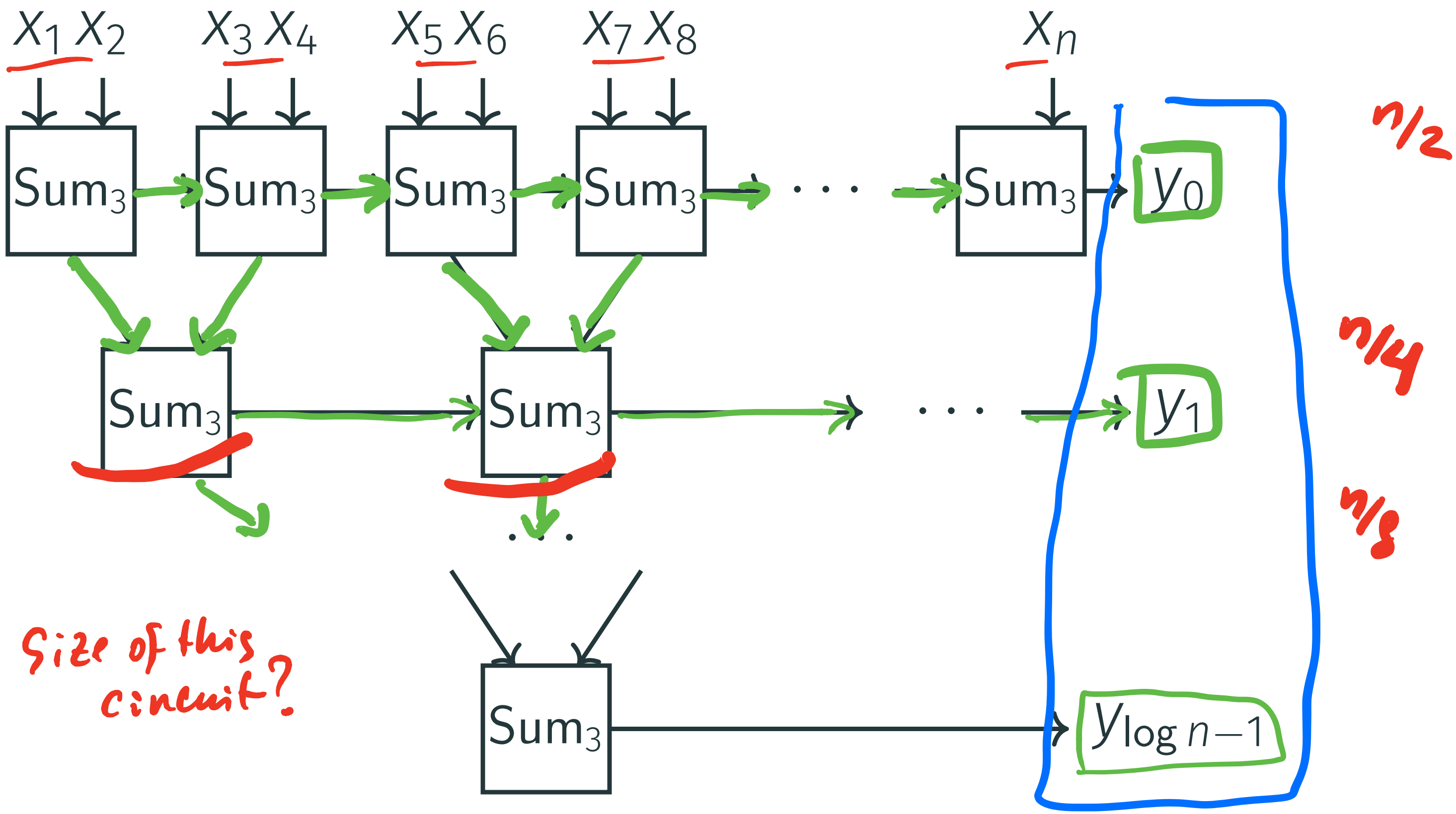
$x_1 + x_2 + x_3 =$
$= 2 \cdot \text{carry} +$
$+ \text{parity}$

Sum$_5$?



$x_1 + x_2 + x_3 + x_4 + x_5$ in binary $y_2\, y_1\, y_0$
$= 4y_2 + 2y_1 + y_0$

Complexity of Sum_n

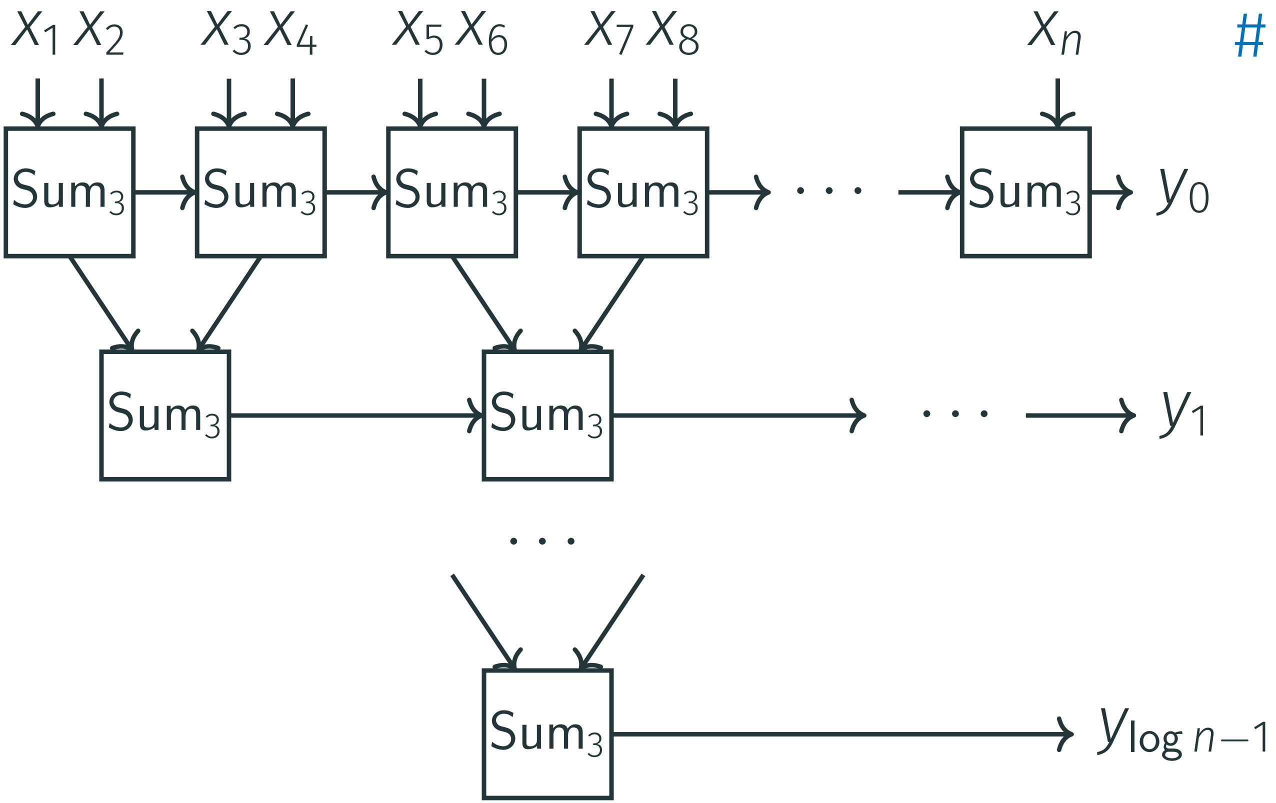


Size of this circuit?

$$x_1 + x_2 + \dots + x_n = y_0 + 2y_1 + 4y_2 + 8y_3 + \dots + 2^{\log n - 1} \cdot y_{\log n - 1}$$

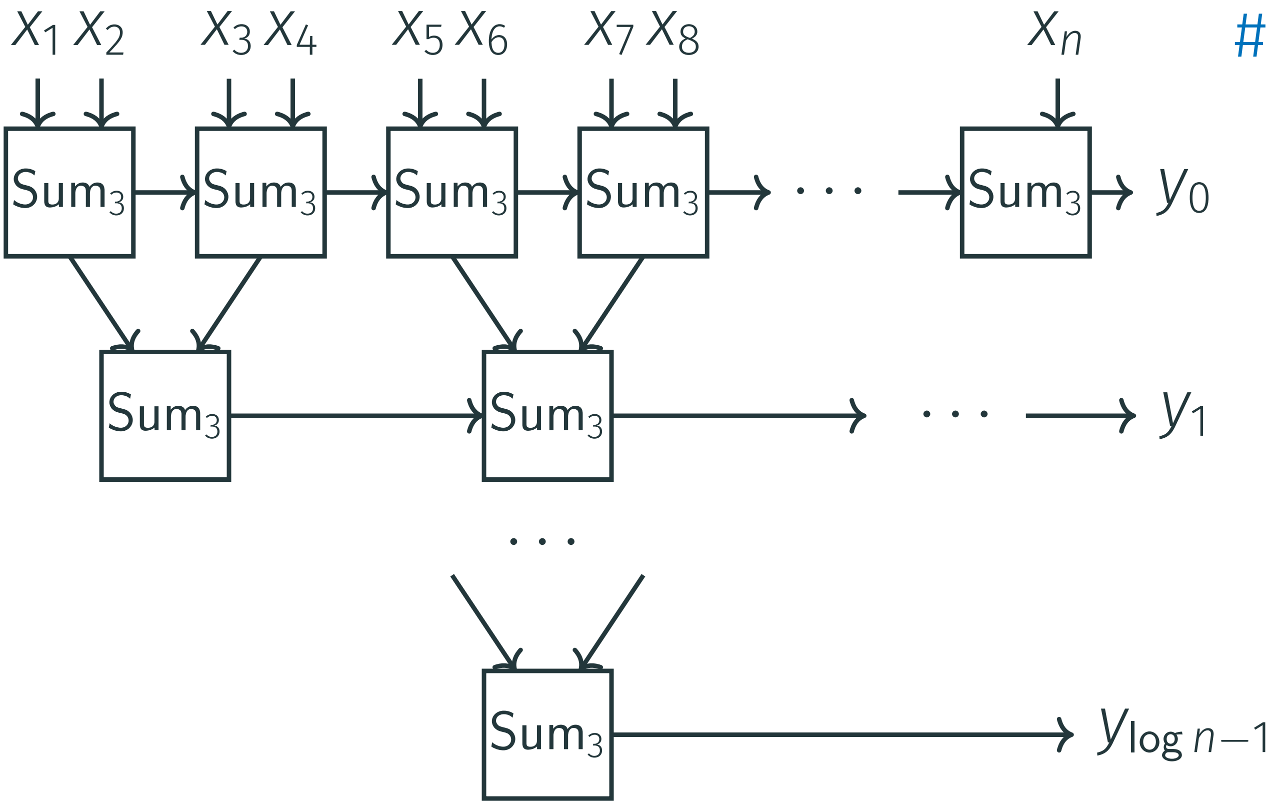
in binary $y_{\log n-1} \dots\dots y_2\, y_1\, y_0$

Complexity of Sum_n



blocks

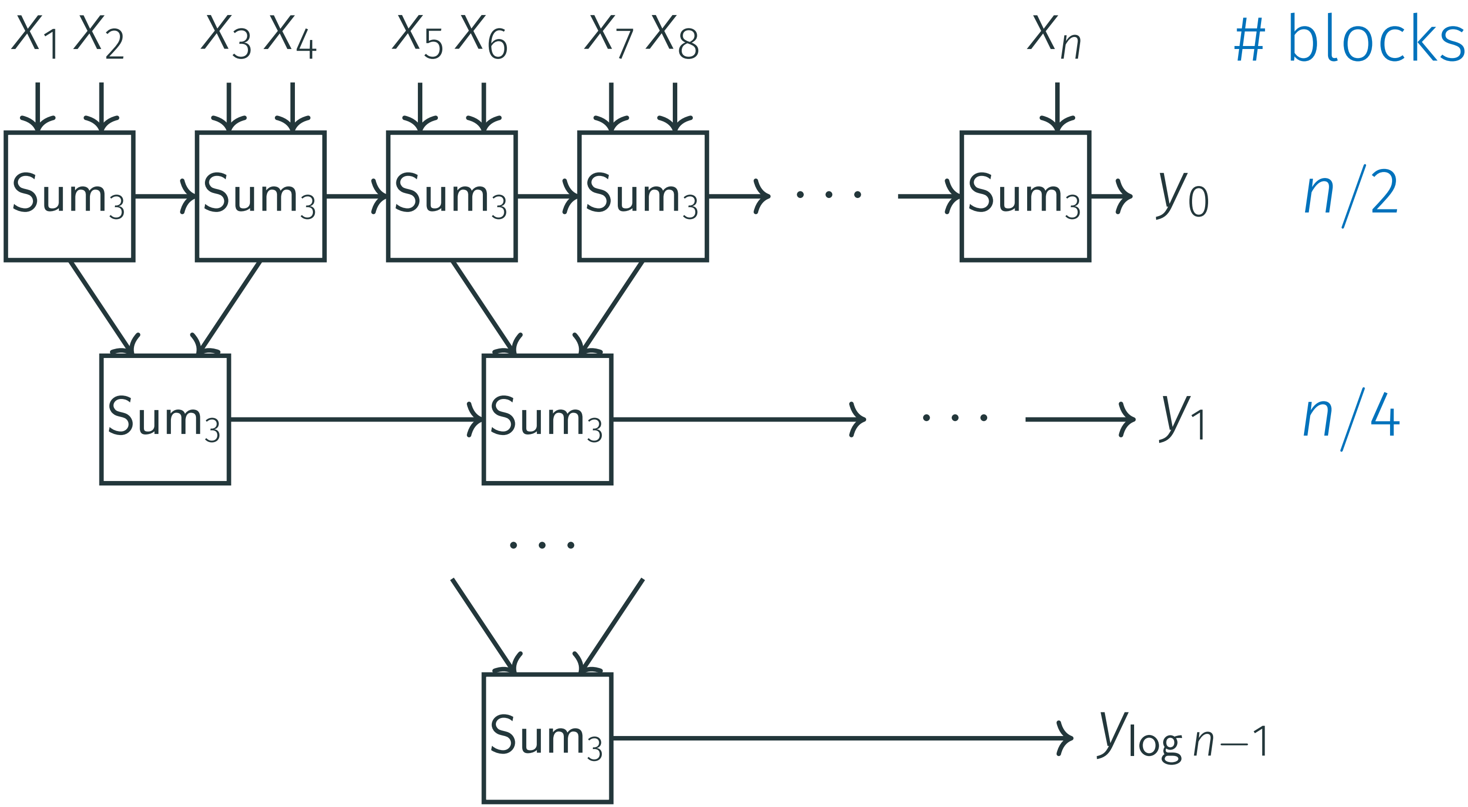
Complexity of Sum_n



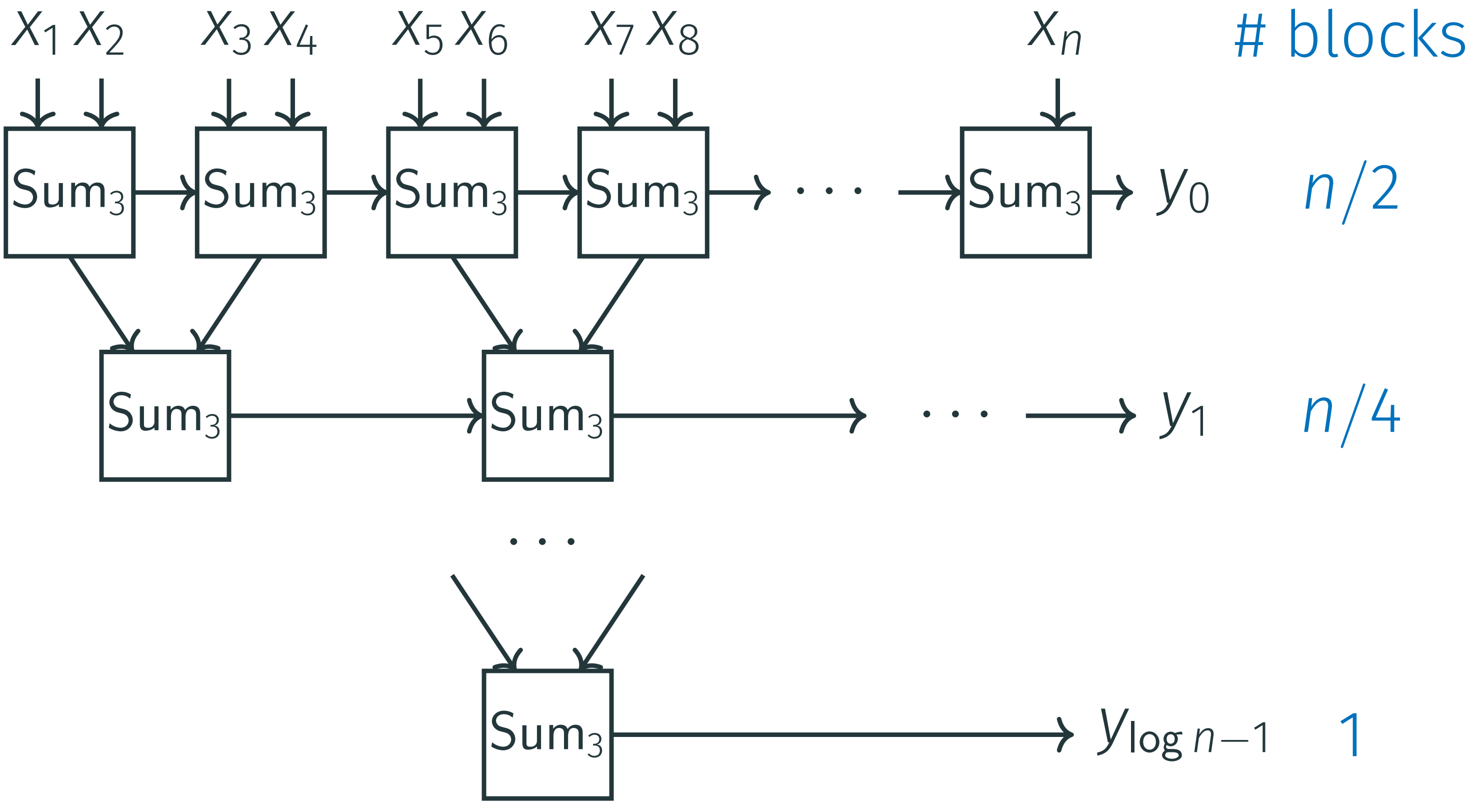
blocks

$n/2$

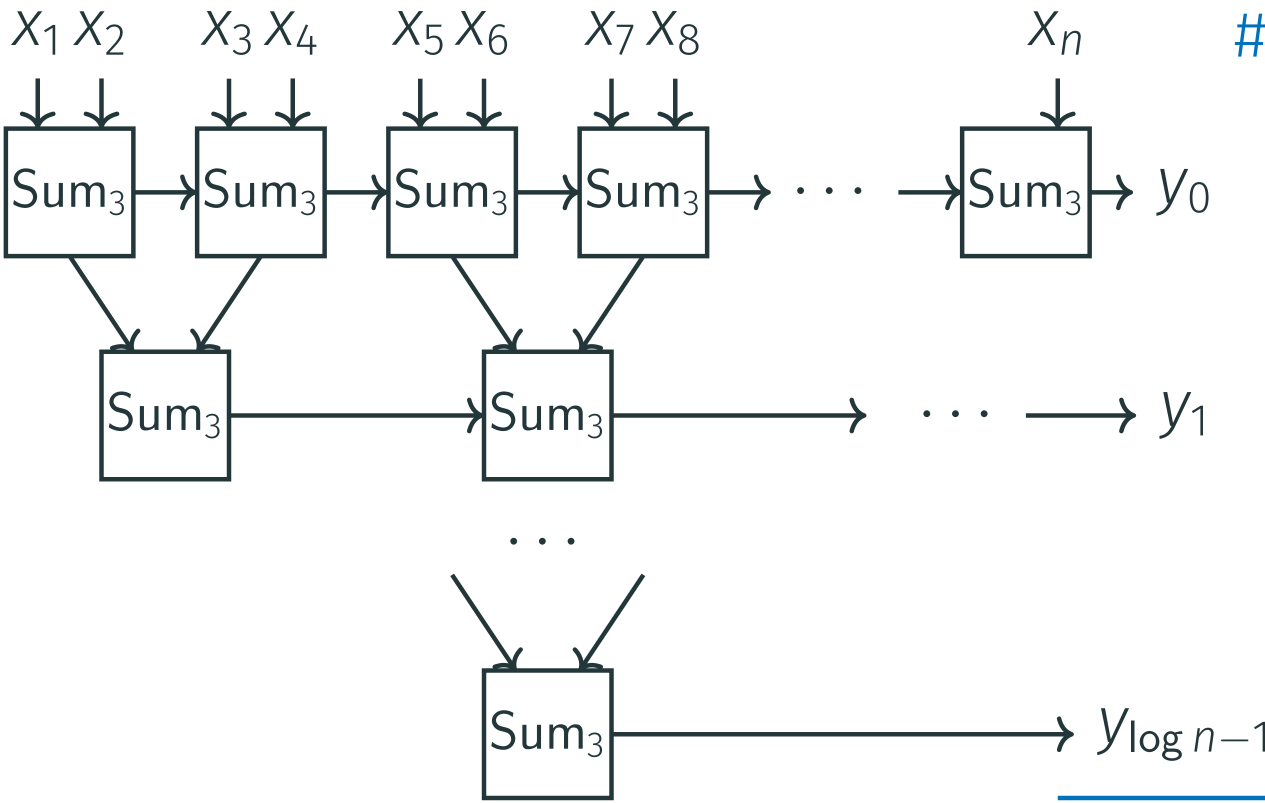
Complexity of Sum_n



Complexity of Sum_n



Complexity of Sum_n

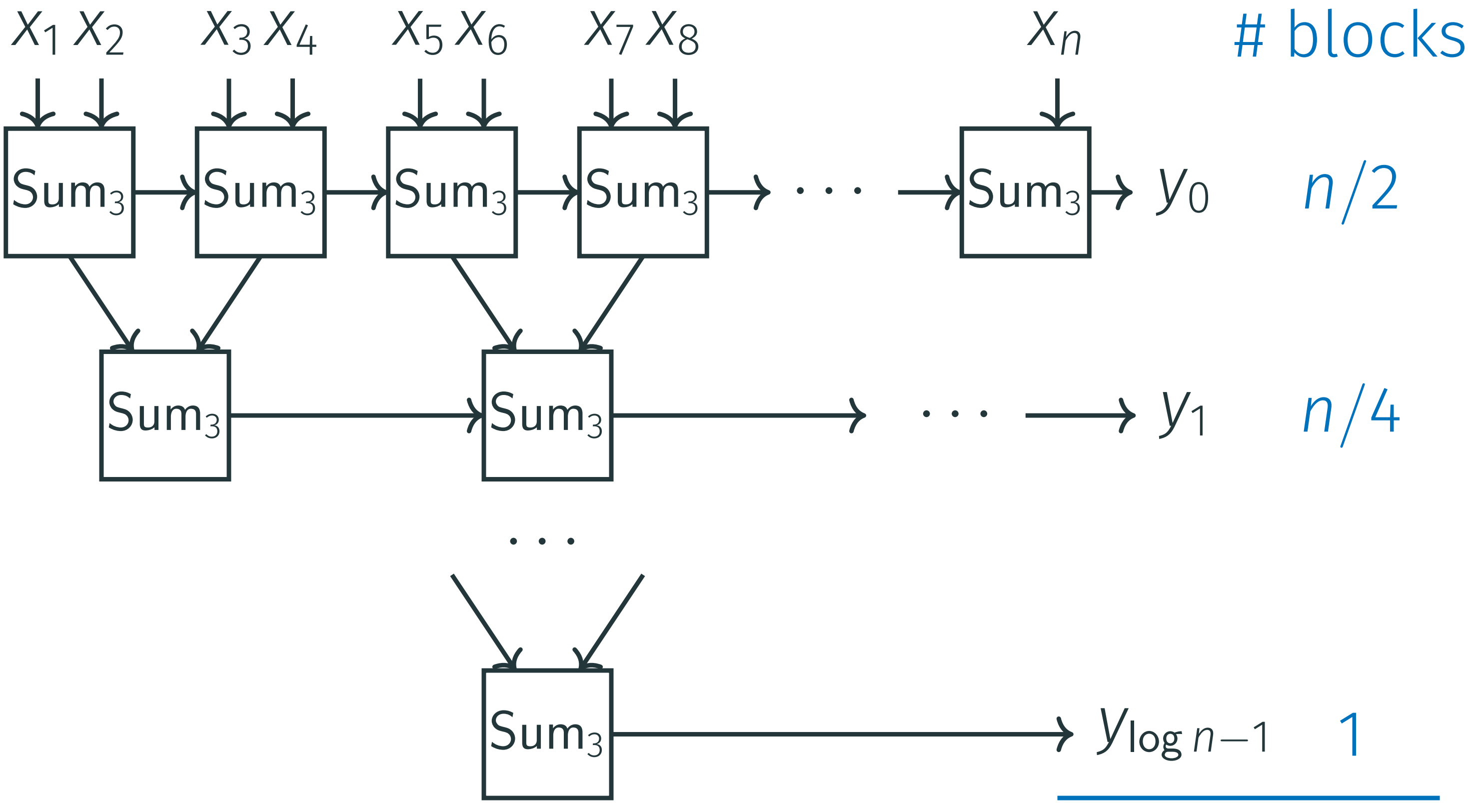


blocks
$n/2$
$n/4$
1

Total: $n-1$

n gadgets Sum_3, each of them is of constant $O(1)$ circuit complexity $\Rightarrow$ circuit complexity is $O(n)$.

Complexity of Sum_n



$\mathrm{Size}(\mathrm{Sum}_n) < n \cdot \mathrm{Size}(\mathrm{Sum}_3) = O(n)$

Total: $n - 1$

Complexity of Symmetric Functions

Theorem

If $f\colon \{0,1\}^n \to \{0,1\}$ is a symmetric function, then

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$O(n)$

$$f = h\left(\boxed{\mathrm{Sum}_n(x_1, \ldots, x_n)}\right), \quad h\colon \{0,1\}^{\log n} \to \{0,1\}$$

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$$\mathrm{Size}(f) \leq \mathrm{Size}(\mathrm{Sum}_n) + \mathrm{Size}(h)$$

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$$f = h(\underbrace{\mathrm{Sum}_n(x_1,\ldots,x_n)}),\ \ h\colon \underbrace{\{0,1\}^{\boxed{\log n}}} \to \{0,1\}$$

Lup: any function $h:\{0,1\}^m \to \{0,1\}$ size$(h) \le$ $\le 10 \cdot 2^m$

$$\mathrm{Size}(f) \le \mathrm{Size}(\mathrm{Sum}_n) + \mathrm{Size}(h)$$

$$\mathrm{Size}(\mathrm{Sum}_n) = O(n)$$

$$\mathrm{Size}(h) \le 10 \cdot 2^{\log n} = \boxed{O(n)}$$

$m = \log n$

$\square$

Complexity of Threshold

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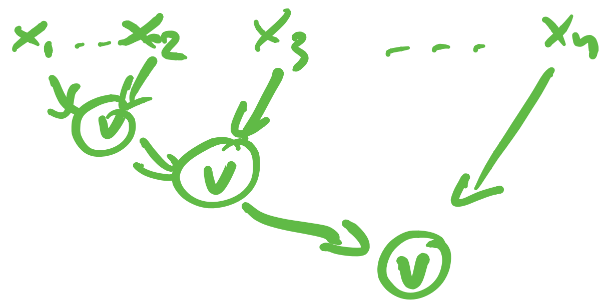

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- $k = 2$

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- Two rounds of “Bubble Sort”

$Th_2(x_1, x_2, \ldots, x_n) = 1$ iff $\sum_{i=1}^{n} x_i \geq 2$

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1	0	0	1	1	0	1	1	0	0

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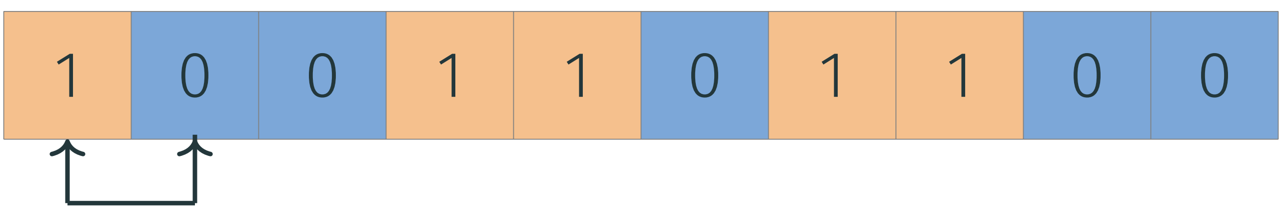

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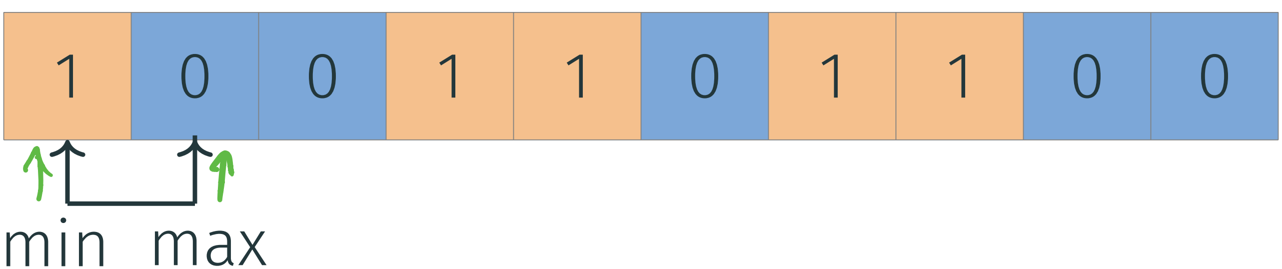

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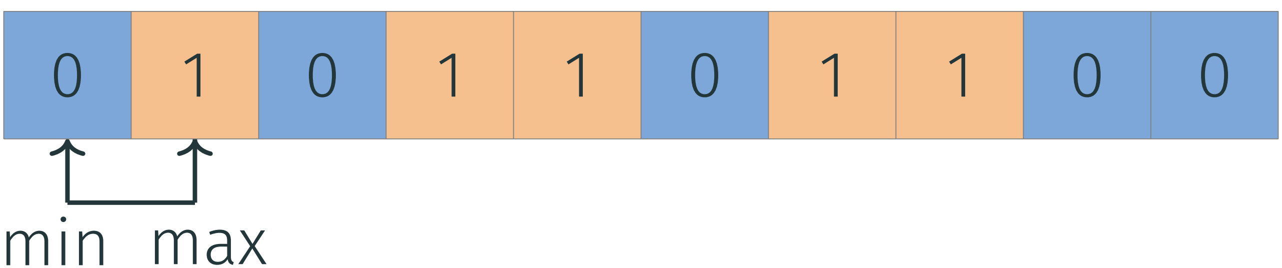

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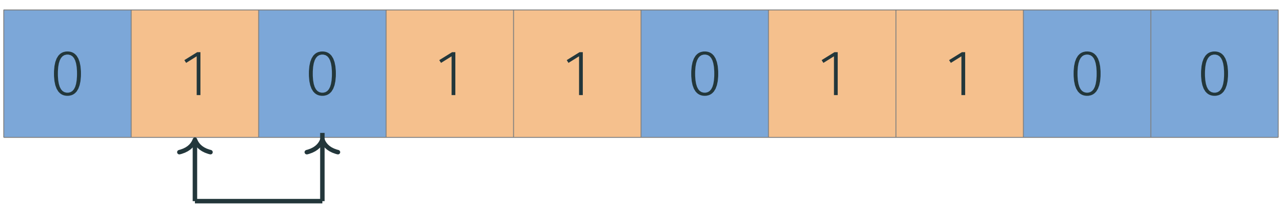

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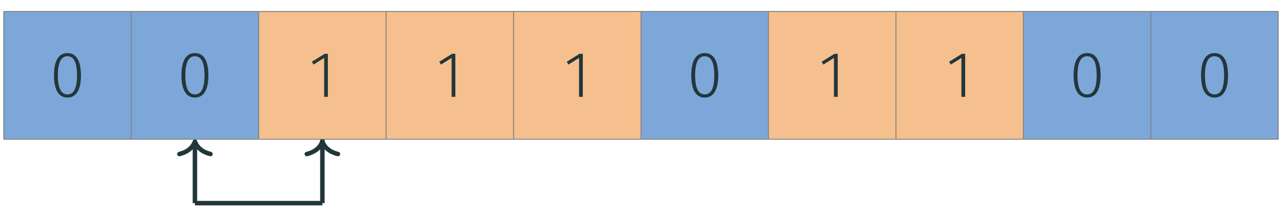

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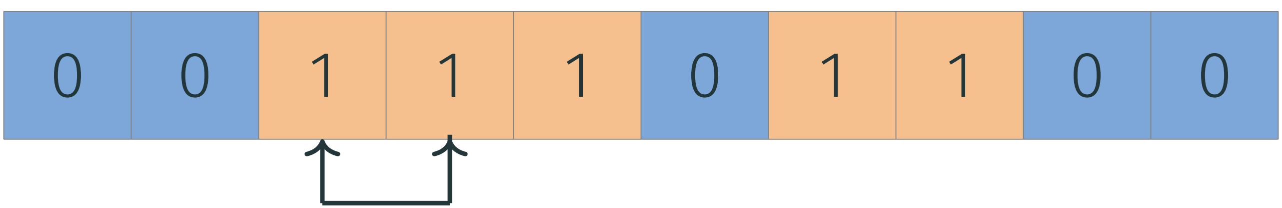
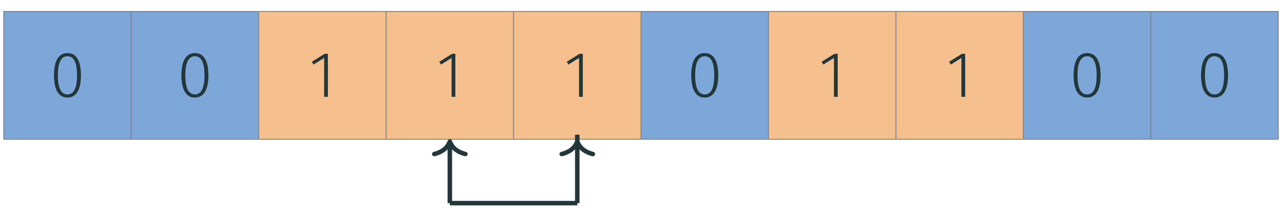
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



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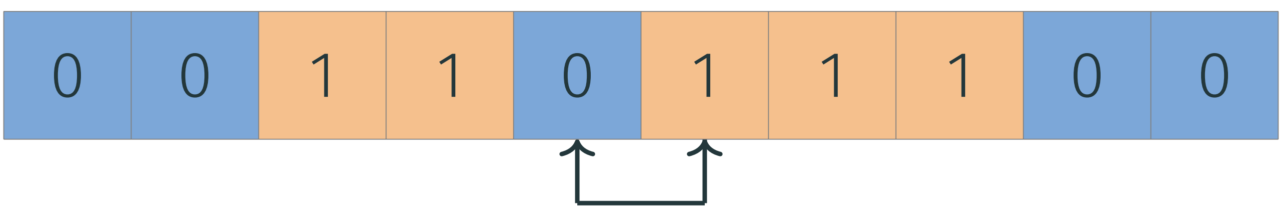

Complexity of Threshold

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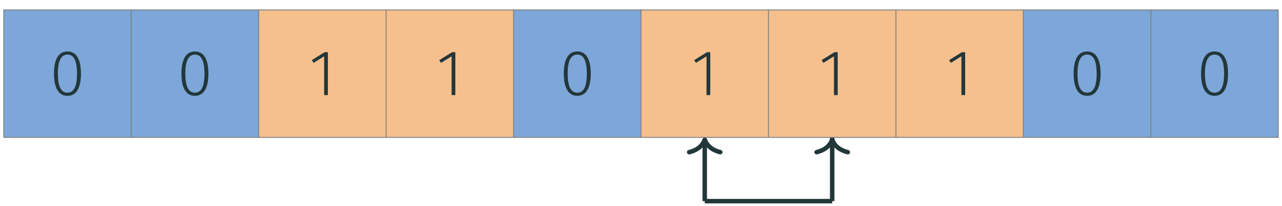

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- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of "Bubble Sort"

0	0	1	1	0	1	1	1	0	0

min max

$\max(x_1, x_2) = x_1 \vee x_2$

$\min(x_1, x_2) = x_1 \wedge x_2$

Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

- $k = 1, \mathrm{Th}_1 = \mathrm{Or}, \mathrm{Size}(\mathrm{Th}_1) = n - 1$
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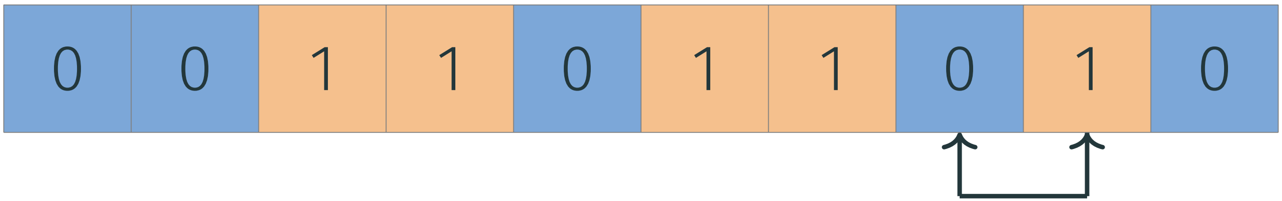

Complexity of Threshold

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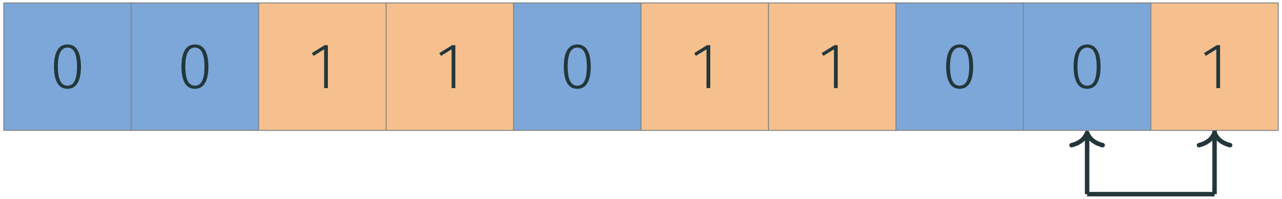

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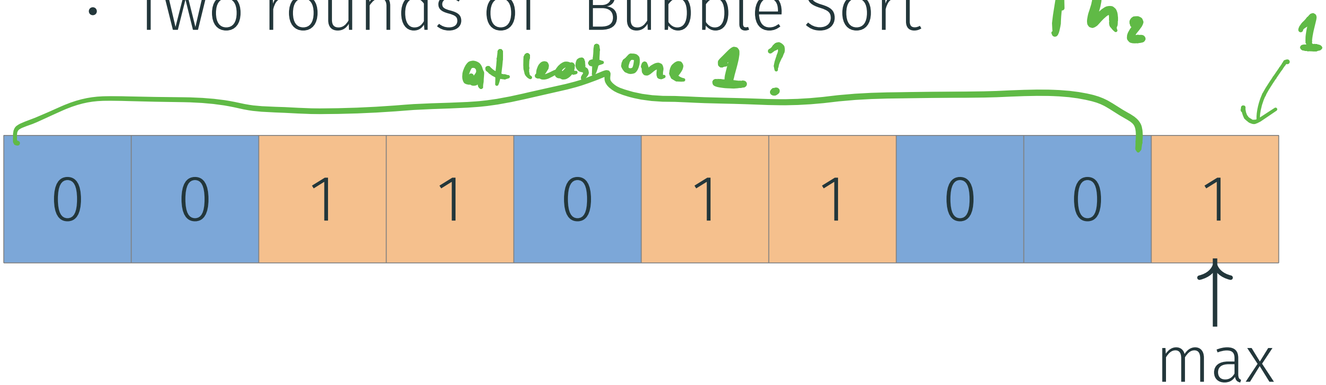
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”

Th_2



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$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

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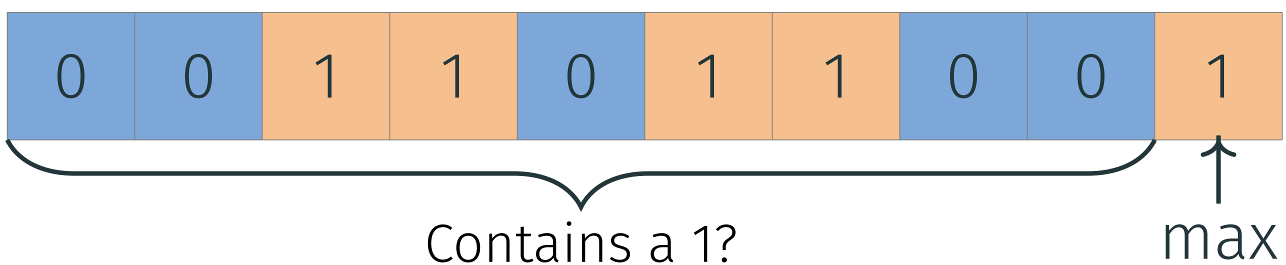
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Complexity of Threshold

$$\mathrm{Th}_k(x) = 1 \text{ iff } x_1 + \ldots + x_n \geq k\,.$$

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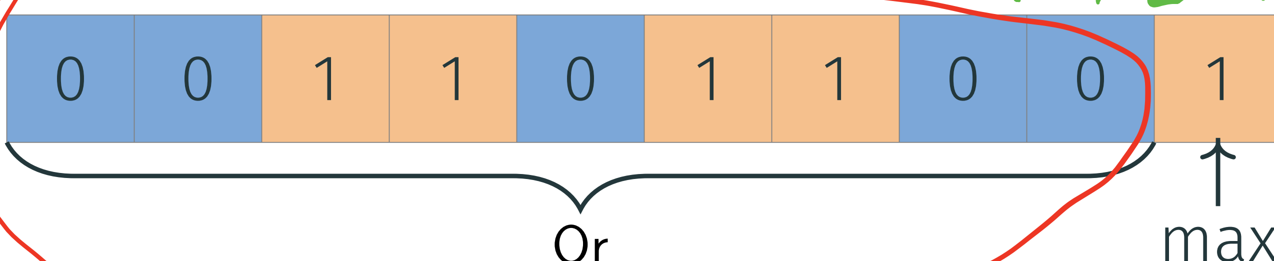
- $k = 2, \mathrm{Size}(\mathrm{Th}_2) = O(n)$
- $\mathrm{Size}(\mathrm{Th}_2) < 3n$
- Two rounds of "Bubble Sort"

size(round of Bubble Sort) $= (n-1) \cdot$ [Compute min & max]



$= 2(n-1)$

size(OR) $< n$

$\Rightarrow$ size $< n + 2(n-1) < 3n$

Th_2. Upper Bound

$\mathrm{Th}_2 \equiv 1$ IF ≥ 2 ones in the input

$x_1, x_2, \ldots, x_n$

$$\sqrt{n}\left\{\overbrace{\begin{array}{|ccc|}\hline X_1 & \cdots & X_{\sqrt{n}} \\ \vdots & & \vdots \\ X_{n-\sqrt{n}+1} & & X_n \\ \hline\end{array}}^{\sqrt{n}}\right.$$

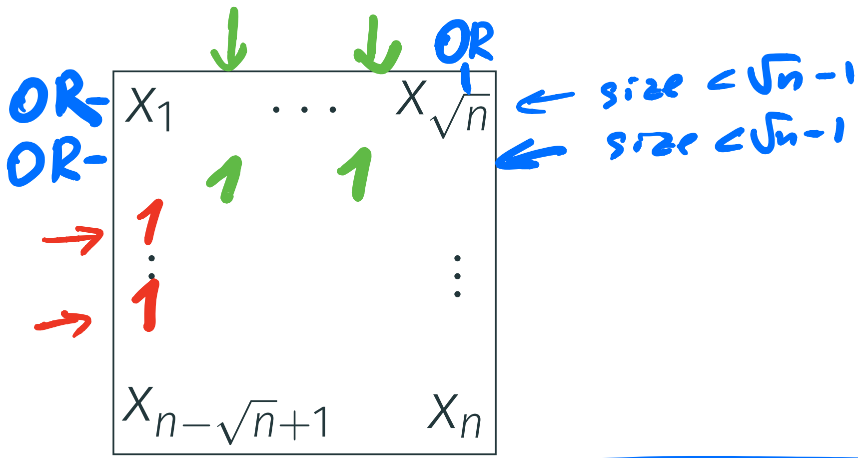
Th_2. Upper Bound

$$\boxed{\begin{array}{ccc} x_1 & \cdots & x_{\sqrt{n}} \\ \vdots & & \vdots \\ x_{n-\sqrt{n}+1} & & x_n \end{array}}$$

$\mathsf{Th}_2(x_1, \ldots, x_n) = 1$ iff

Th_2. Upper Bound

size(OR_m) $= m-1$



$\mathrm{Th}_2(x_1, \ldots, x_n) = 1$ iff

there are ≥ two cols with 1s

OR

there are ≥ two rows with 1s

Th_2. Upper Bound

size $\le \sqrt{n}\cdot(\sqrt{n}-1) < n$

$$\begin{array}{ll|ccc|}
\cline{3-5}
\sqrt{n}-1 & y_1 = \mathrm{Or} & x_1 & \cdots & x_{\sqrt{n}} \\
\sqrt{n}-1 & y_2 = \mathrm{Or} & & & \\
 & \vdots & \vdots & & \vdots \\
\sqrt{n}-1 & y_{\sqrt{n}} = \mathrm{Or} & x_{n-\sqrt{n}+1} & & x_n \\
\cline{3-5}
\end{array}$$

there are two cols with 1s

$\mathrm{Th}_2(x_1, \dots, x_n) = 1$ iff

OR

there are two rows with 1s

Th_2. Upper Bound

size < n gates

$$\begin{array}{r|cccc|} & z_1 = \text{Or} & z_2 = \text{Or} & \cdots & z_{\sqrt{n}} = \text{Or} \\ \hline y_1 = \text{Or} & x_1 & & \cdots & x_{\sqrt{n}} \\ y_2 = \text{Or} & & & & \\ \vdots & \vdots & & & \vdots \\ y_{\sqrt{n}} = \text{Or} & x_{n-\sqrt{n}+1} & & & x_n \\ \hline \end{array}$$

$\text{Th}_2(x_1, \ldots, x_n) = 1$ iff

there are two cols with 1s

OR

there are two rows with 1s

Th_2. Upper Bound

$$
\begin{array}{rl|ccc|}
 & & z_1 = \mathrm{Or} \quad z_2 = \mathrm{Or} & \cdots & z_{\sqrt{n}} = \mathrm{Or} \\
\hline
y_1 = & \mathrm{Or} & x_1 & \cdots & x_{\sqrt{n}} \\
y_2 = & \mathrm{Or} & & & \\
\vdots & & \vdots & & \vdots \\
y_{\sqrt{n}} = & \mathrm{Or} & x_{n-\sqrt{n}+1} & & x_n \\
\hline
\end{array}
$$

to compute $y_1 \dots y_{\sqrt{n}}$
$z_1 \dots z_{\sqrt{n}}$
I used $< 2n$ gates

$$\mathrm{Th}_2(x_1, \dots, x_n) = \boxed{\mathrm{Th}_2(y_1, \dots, y_{\sqrt{n}})} \ \mathrm{Or} \ \boxed{\mathrm{Th}_2(z_1, \dots, z_{\sqrt{n}})}$$

≥ 2 rows with 1.

≥ 2 cols with 1

Th$_2$. Upper Bound

$$\begin{array}{r|cccc|}
 & z_1 = \mathrm{Or} & z_2 = \mathrm{Or} & \cdots & z_{\sqrt{n}} = \mathrm{Or} \\ \hline
y_1 = \mathrm{Or} & x_1 & & \cdots & x_{\sqrt{n}} \\
y_2 = \mathrm{Or} & & & & \\
\vdots & \vdots & & & \vdots \\
y_{\sqrt{n}} = \mathrm{Or} & x_{n-\sqrt{n}+1} & & & x_n \\ \hline
\end{array}$$

Th$_2$ is symmetric function

Th$_2$ with m inputs can be solved by a circuit of size $O(m)$

$$\mathrm{Th}_2(x_1, \ldots, x_n) = \mathrm{Th}_2(y_1, \ldots, y_{\sqrt{n}}) \ \mathrm{Or} \ \mathrm{Th}_2(z_1, \ldots, z_{\sqrt{n}})$$

$$\mathrm{Size}(\mathrm{Th}_2(n)) \le 2n + \underbrace{\boxed{2\,\mathrm{Size}(\mathrm{Th}_2(\sqrt{n}))}}_{= 2O(\sqrt{n})}$$

Th_2. Upper Bound

$$
\begin{array}{rc|ccc|}
 & & z_1 = \mathrm{Or} & z_2 = \mathrm{Or} \quad \cdots & z_{\sqrt{n}} = \mathrm{Or} \\
\hline
y_1 = \mathrm{Or} & & x_1 & \cdots & x_{\sqrt{n}} \\
y_2 = \mathrm{Or} & & & & \\
\vdots & & \vdots & & \vdots \\
y_{\sqrt{n}} = \mathrm{Or} & & x_{n-\sqrt{n}+1} & & x_n \\
\hline
\end{array}
$$

1. Some functions
   size $\geq \frac{2^n}{n}$
2. Symmetric functions
   size $\leq O(n)$
3. Th_2
   size $\leq 2n + o(n)$

$$\mathrm{Th}_2(x_1, \ldots, x_n) = \mathrm{Th}_2(y_1, \ldots, y_{\sqrt{n}}) \ \mathrm{Or}\ \mathrm{Th}_2(z_1, \ldots, z_{\sqrt{n}})$$

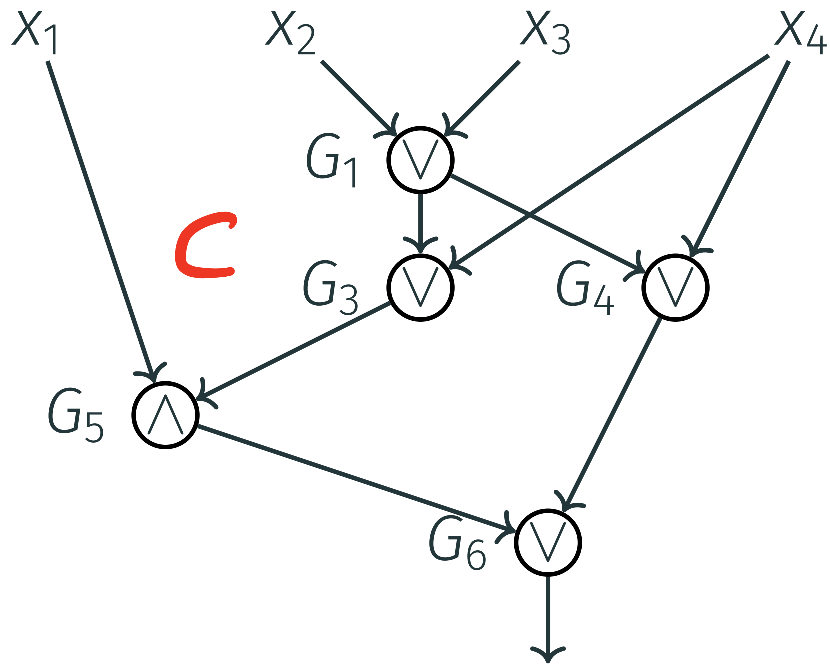
$$\mathrm{Size}(\mathrm{Th}_2(n)) \leq 2n + 2\,\mathrm{Size}(\mathrm{Th}_2(\sqrt{n})) \leq 2n + O(\sqrt{n})$$

$= 2n + o(n)$

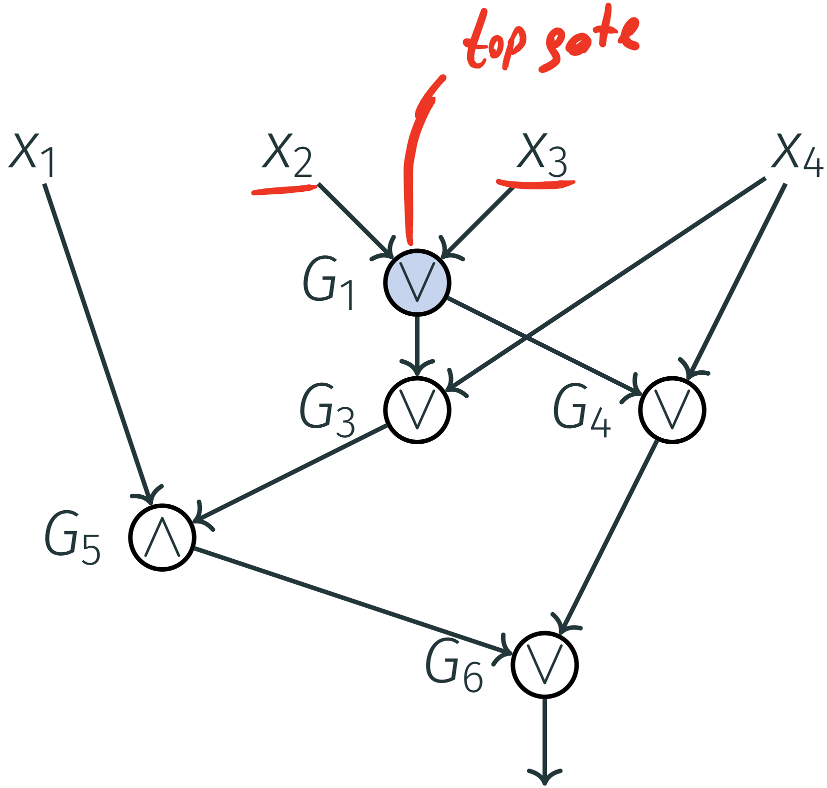
$\boxed{Th_2}$ Lower Bound

If ckt C computes Th_2

$\Rightarrow$ size$(C) \geq 2n - O(1)$



Th_2. Lower Bound



Th$_2$. Lower Bound

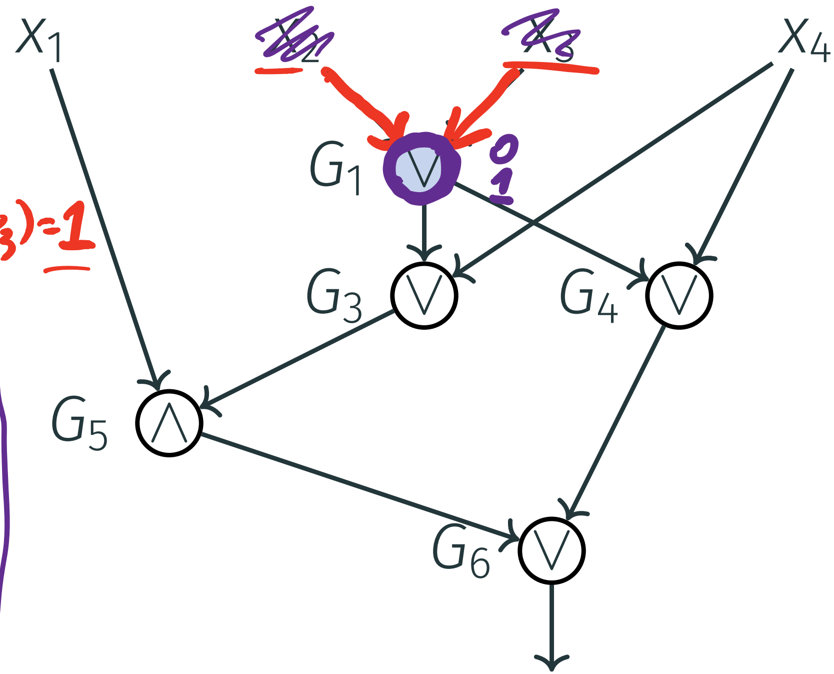
Case I:

Out-deg(x_2) = out-deg(x_3) = 1

Prove impossible

$$\boxed{\begin{array}{l} C(x_1, 0, 0, x_4, \ldots, x_n) \\ C(x_1, 0, 1, x_4, \ldots, x_n) \\ C(x_1, 1, 0, x_4, \ldots, x_n) \\ C(x_1, 1, 1, x_4, \ldots, x_n) \end{array}}$$

two different funs computed by these 4 circuits



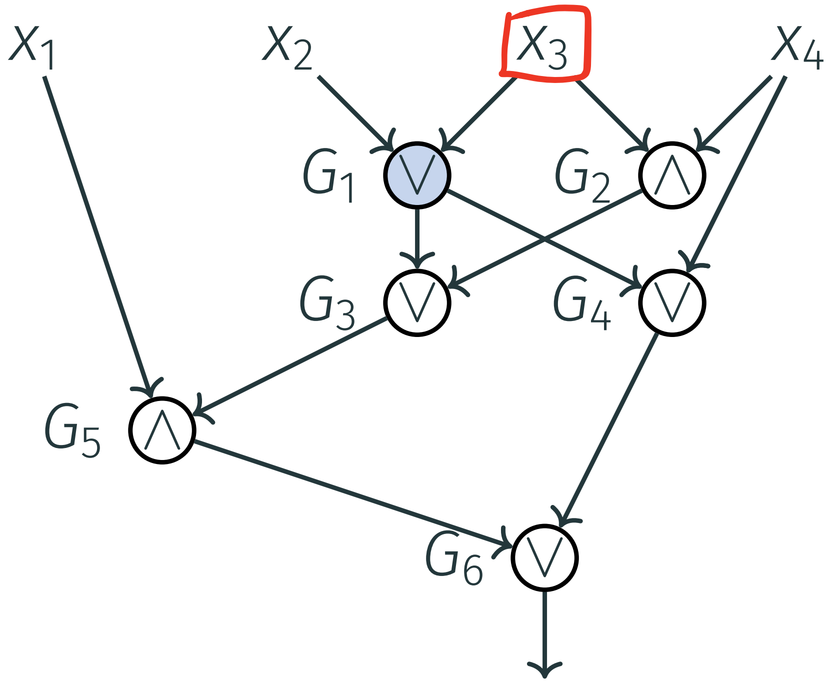
$$\begin{cases} Th_2(x_1, 0, 0, x_4, \ldots, x_n) = \underline{Th_2(x_1, x_4, \ldots, x_n)} \\ Th_2(x_1, 0, 1, x_4, \ldots, x_n) = \\ Th_2(x_1, 1, 0, x_4, \ldots, x_n) = \end{cases} \underline{OR(x_1, x_4, \ldots, x_n)}$$

$$\boxed{Th_2(x_1, 1, 1, x_4, \ldots, x_n) = \underline{1}}$$

Th$_2$. Lower Bound

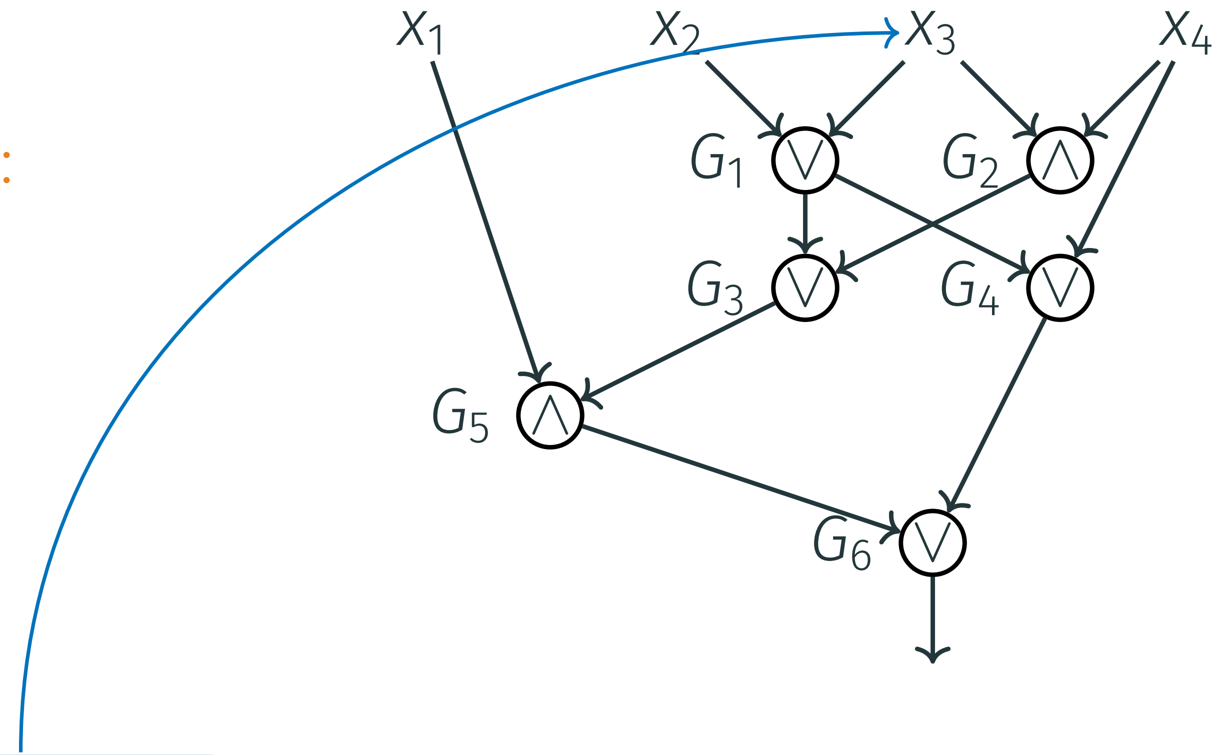
Case II:

out-deg$(x_3) \geq 2$



Th_2. Lower Bound

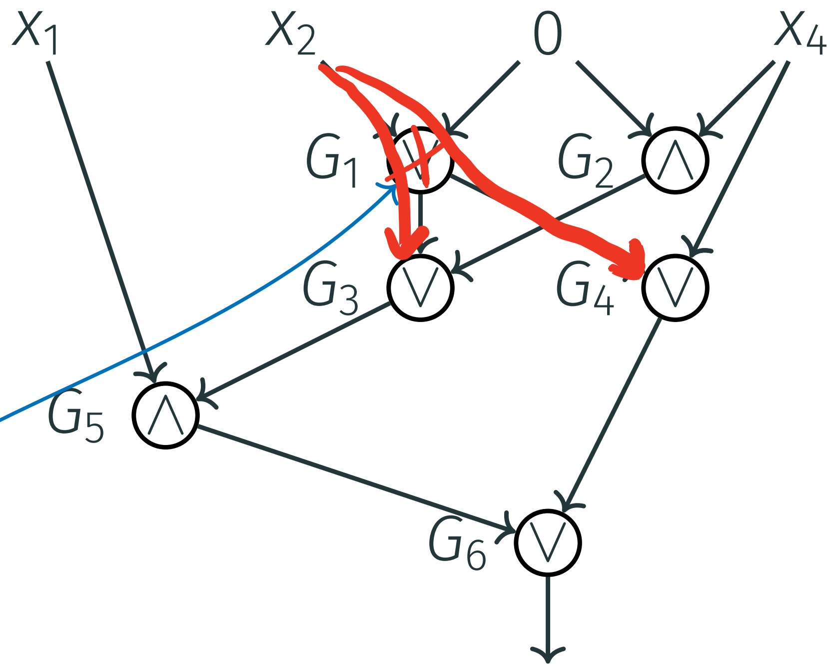
Case II:



assign $x_3 = 0$

Th_2. Lower Bound

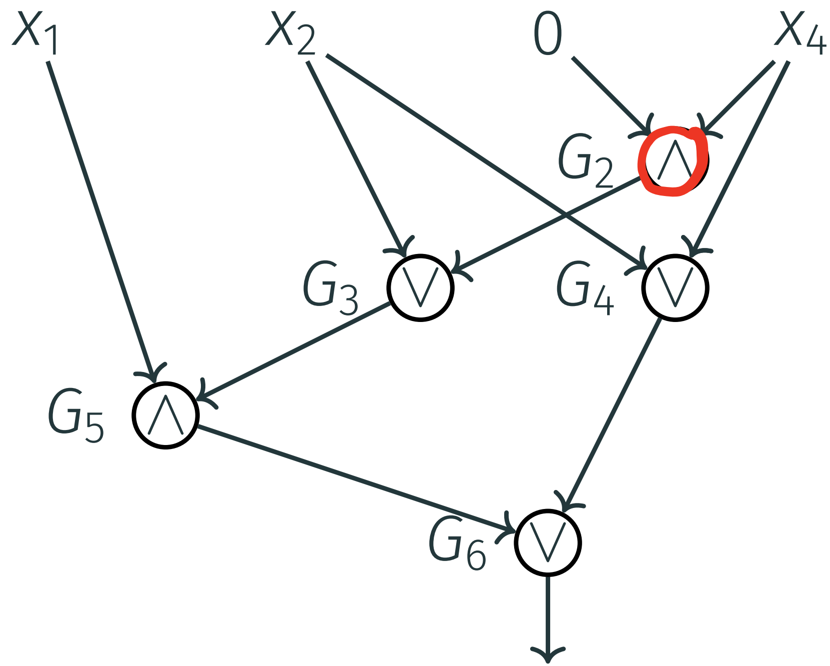
Case II:



G_1 now computes x_2

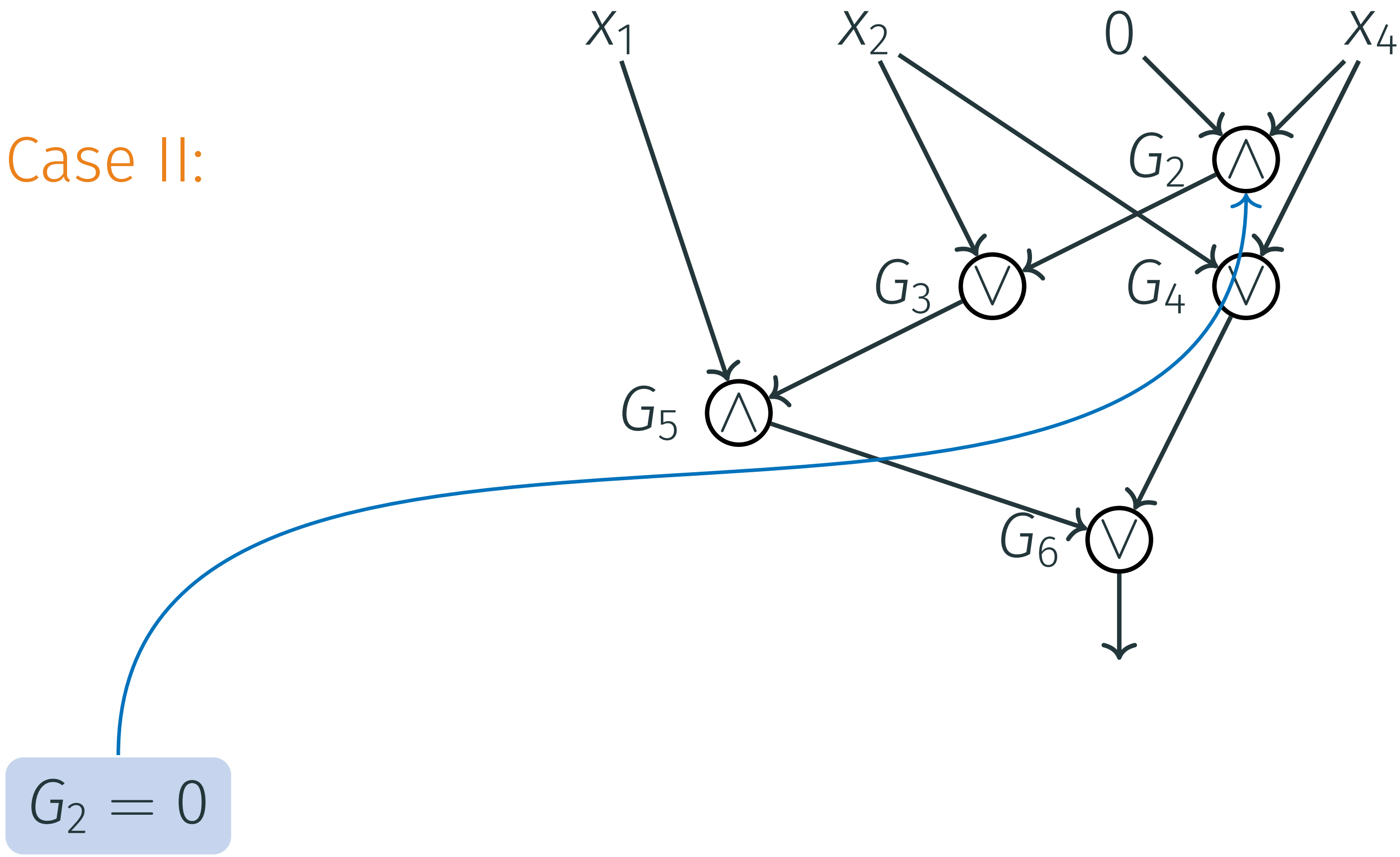
Th_2. Lower Bound

Case II:



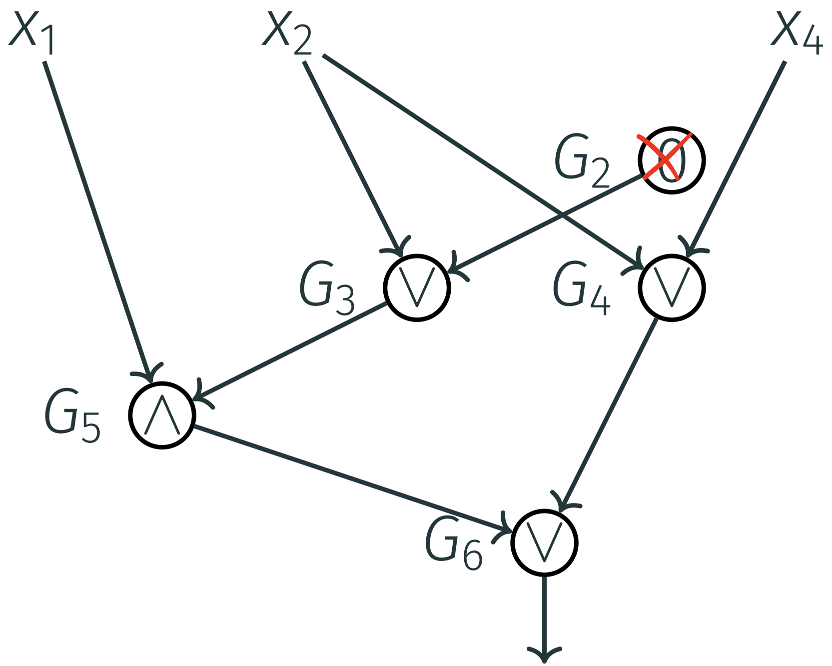
Th_2. Lower Bound

Case II:



Th_2. Lower Bound

Case II:



Th_2. Lower Bound

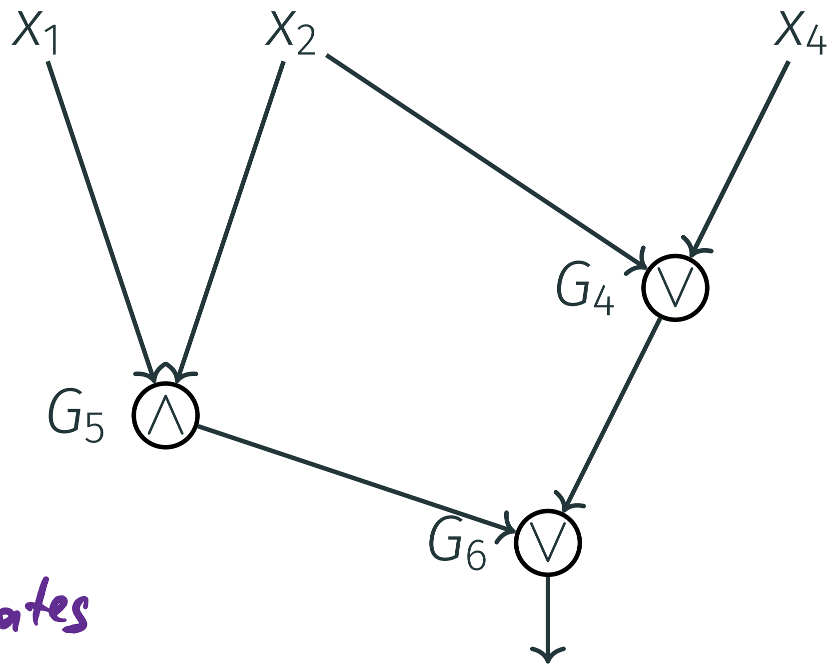
Case II:

out-deg$(x_3) \geq 2$

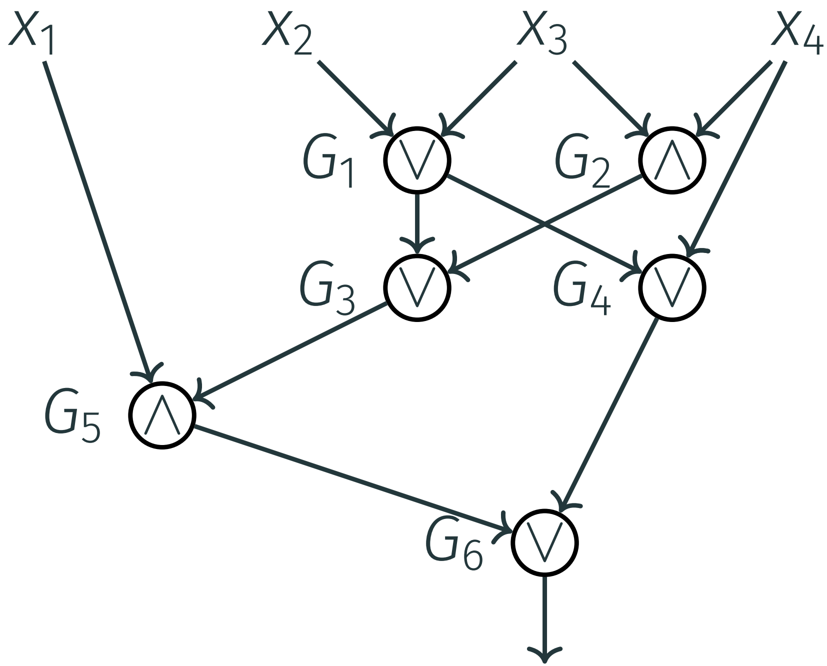
$x_3 = 0$

$\wedge$ $\vee$

Eliminate these two gates



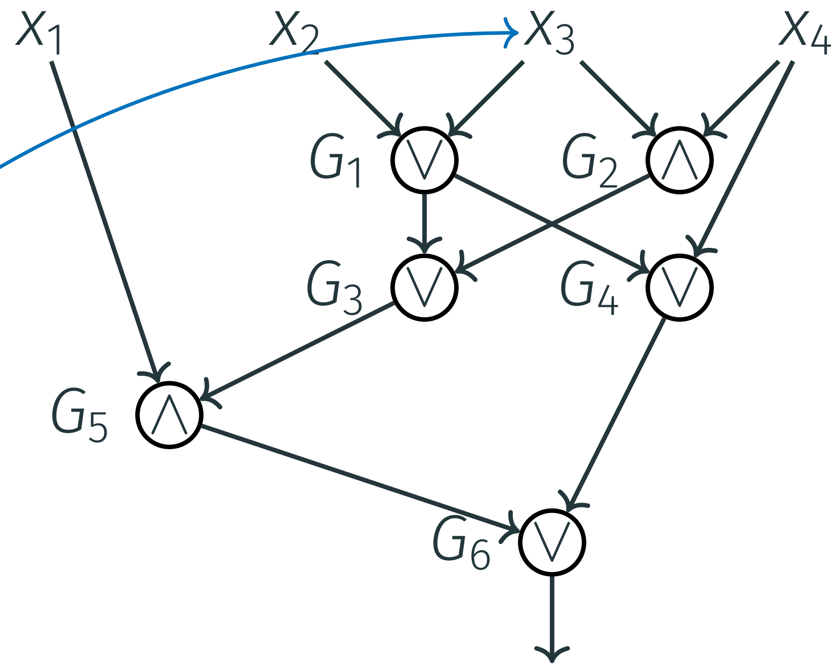
Th_2. Lower Bound



n inputs

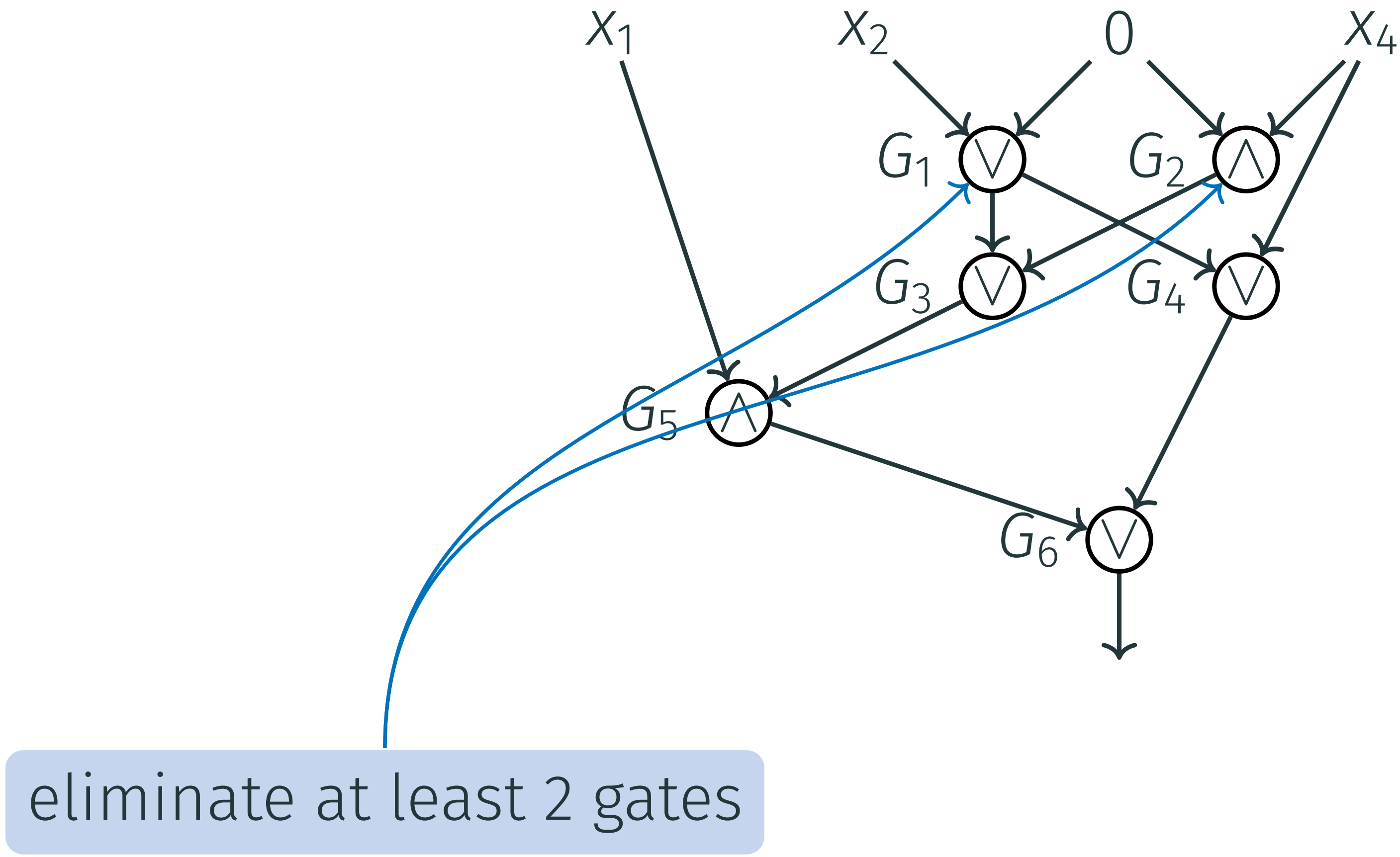
we start with circuit for $\mathbf{Th}_2^n$

Th$_2$. Lower Bound

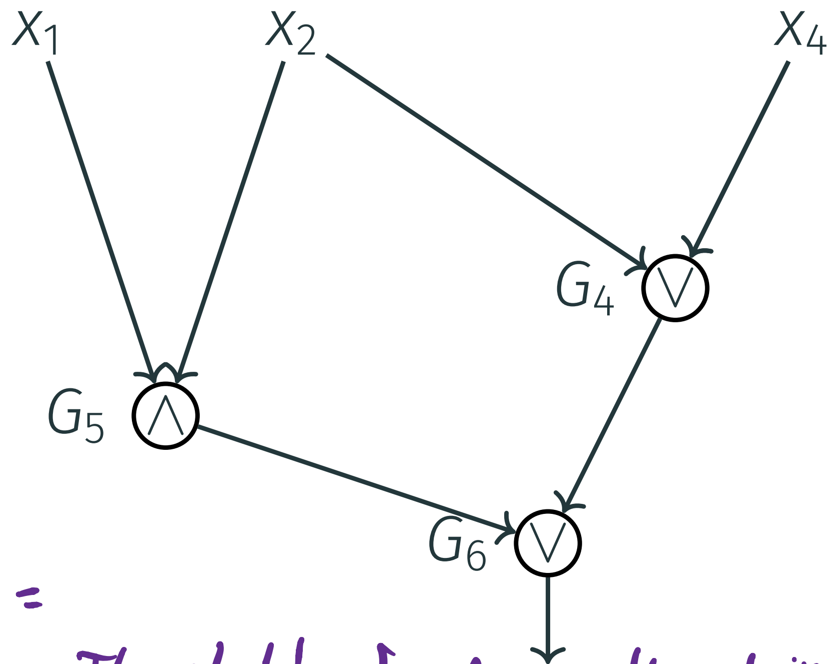


assign $x_3 = 0$

Th$_2$. Lower Bound



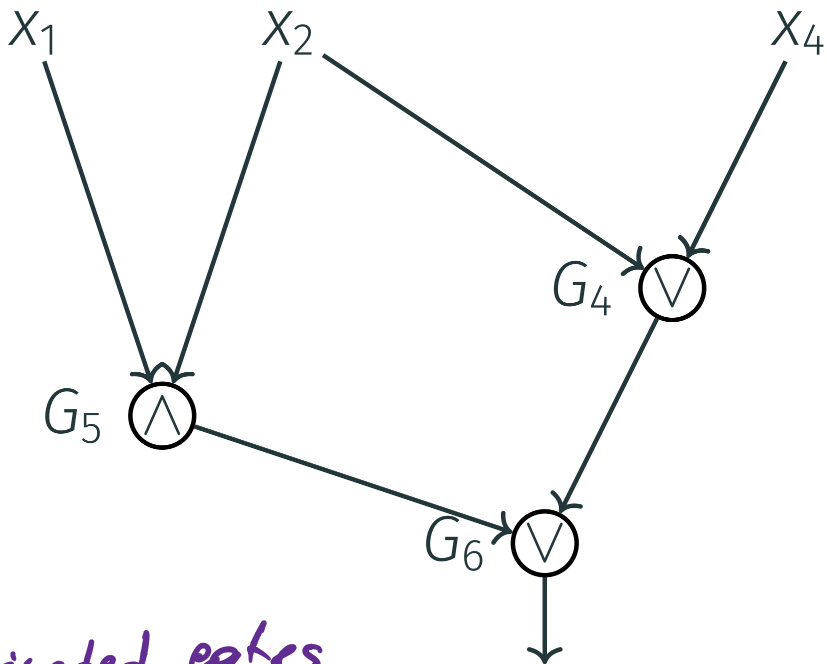
Th_2. Lower Bound



$\mathrm{Th}_2(x_1, x_2, 0, x_4, \dots, x_n) =$
$= \mathrm{Th}_2(x_1, x_2, x_4, \dots, x_n)$ — Threshold$_2$ function with n-1 inputs

get a circuit for $\mathbf{Th}_2^{n-1}$

Th_2. Lower Bound

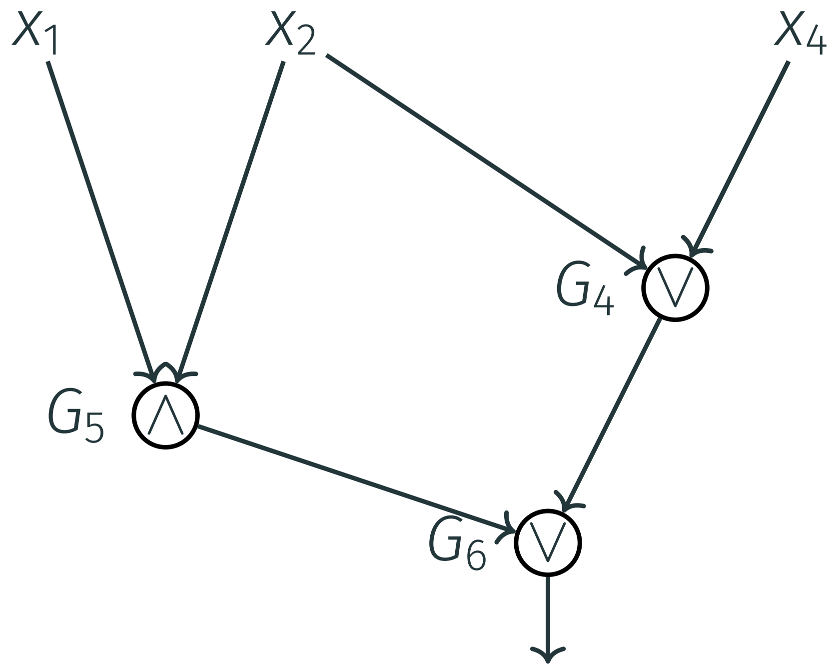


2 eliminated gates

$$\boxed{\mathrm{Size}(\mathrm{Th}_2^n)} \geq \boxed{2} + \mathrm{Size}(\mathrm{Th}_2^{n-1}) \geq 2+2+\mathrm{Size}(\mathrm{Th}_2^{n-2}) \geq$$

$$\ldots \geq 2n - O(1)$$

Th_2. Lower Bound



$$\mathrm{Size}(\mathrm{Th}_2^n) \geq 2 + \mathrm{Size}(\mathrm{Th}_2^{n-1}) \geq 2n - O(1)$$

$\mathrm{size}(\mathrm{Th}_2^n) = 2n \pm o(n)$

$\square$