Gems of TCS

Graph Coloring Algorithms

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February 18, 2021

Previously...

- Exact Algorithms

Exp-time alg

- Randomized Algorithms
- Approximate Algorithms

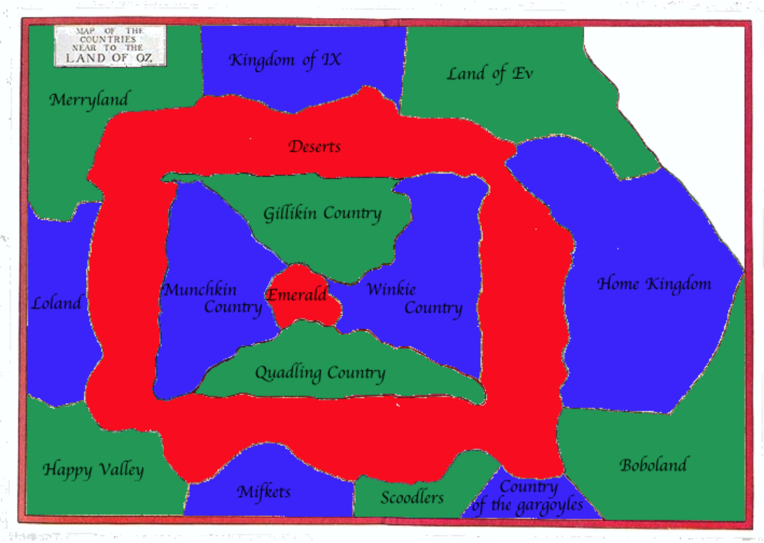
Previously...

- Exact Algorithms
- Randomized Algorithms
- Approximate Algorithms
- Today: More examples

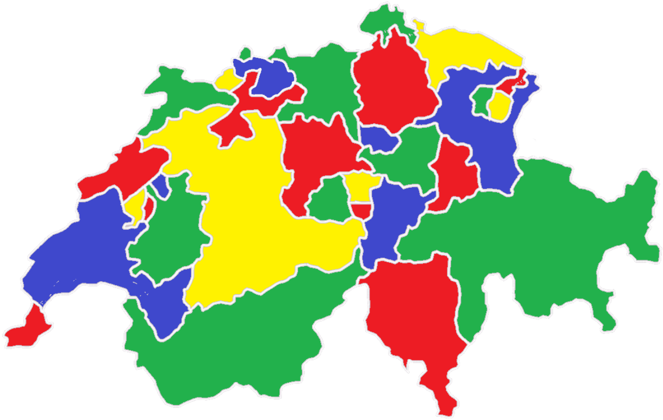
Map Coloring

South America

The Land of Oz



Swiss Cantons



Four Color Theorem

Theorem (Appel, Haken, 1976) ✗

Every map can be colored with 4 colors.

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Four Color Theorem

Theorem (Appel, Haken, 1976)

Every map can be colored with 4 colors.

- Proved using a computer.
- Computer checked almost 2000 graphs.
- Robertson, Sanders, Seymour, and Thomas gave a much simpler proof in 1997 (still using a computer search).

Theorem (Weak Version)

Every map can be colored with 6 colors.

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- Base case. $n \leq 6$: can color with 6 colors.
- Induction assumption. All maps with k countries can be colored with 6 colors.

Six Color Theorem

Theorem (Weak Version)

Every map can be colored with 6 colors.

- Induction on the number of countries n.
- Base case. $n \leq 6$: can color with 6 colors.
- Induction assumption. All maps with k countries can be colored with 6 colors.
- Induction step. We'll show that any map with $k+1$ countries can be colored with 6 colors.

Six Color Theorem. Proof

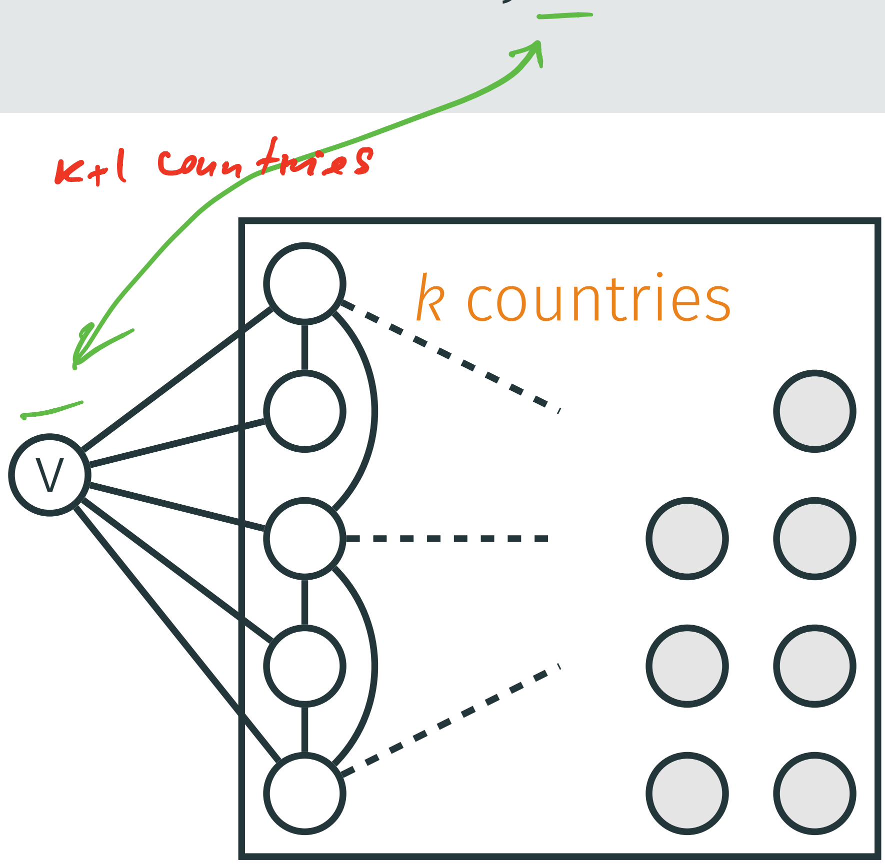
Lemma Euler's Flg for planar graphs

Every map contains a country v with at most 5 neighbors.

Six Color Theorem. Proof

Lemma

Every map contains a country v with at most 5 neighbors.



Six Color Theorem. Proof

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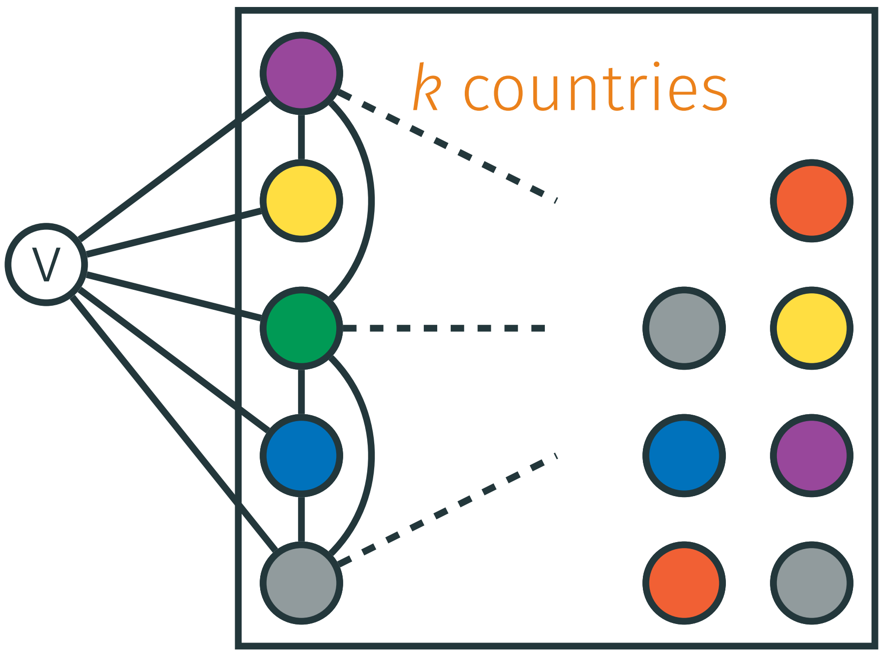

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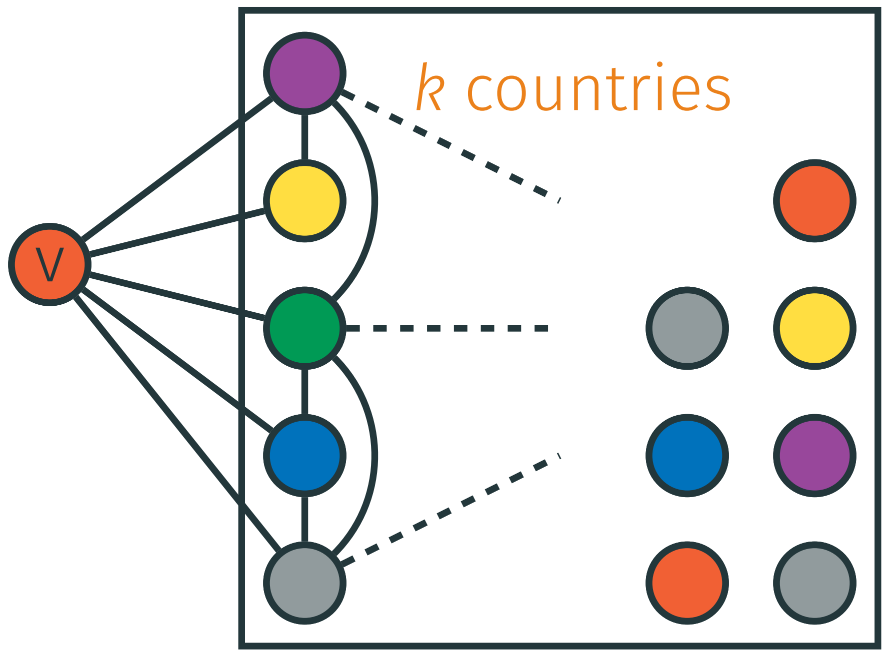
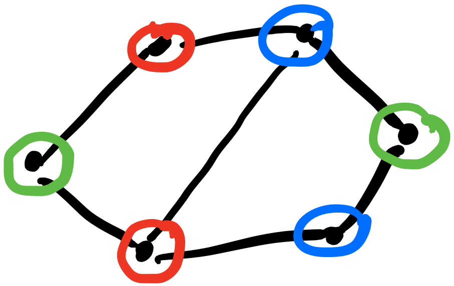


Graph Coloring

Graph Coloring

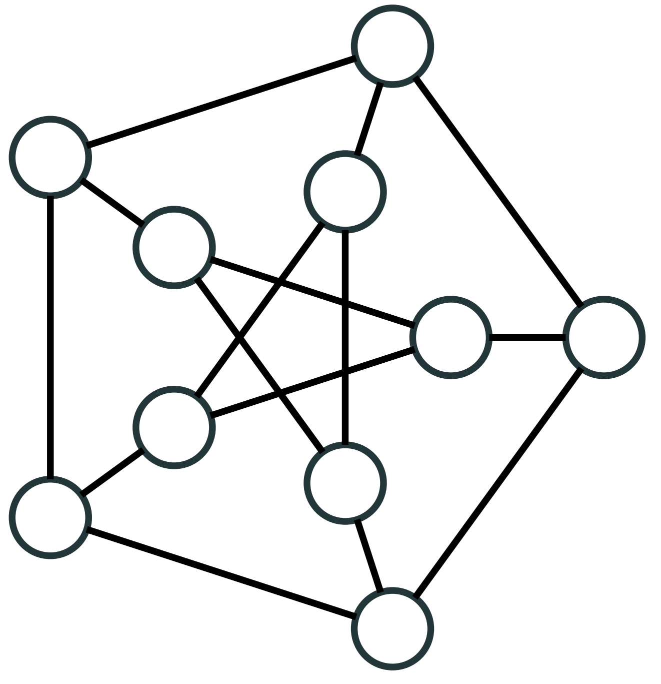
- A graph coloring is a coloring of the graph vertices s.t. no pair of adjacent vertices share the same color.



Graph Coloring

- A graph coloring is a coloring of the graph vertices s.t. no pair of adjacent vertices share the same color.
- The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the graph.

Chromatic Number



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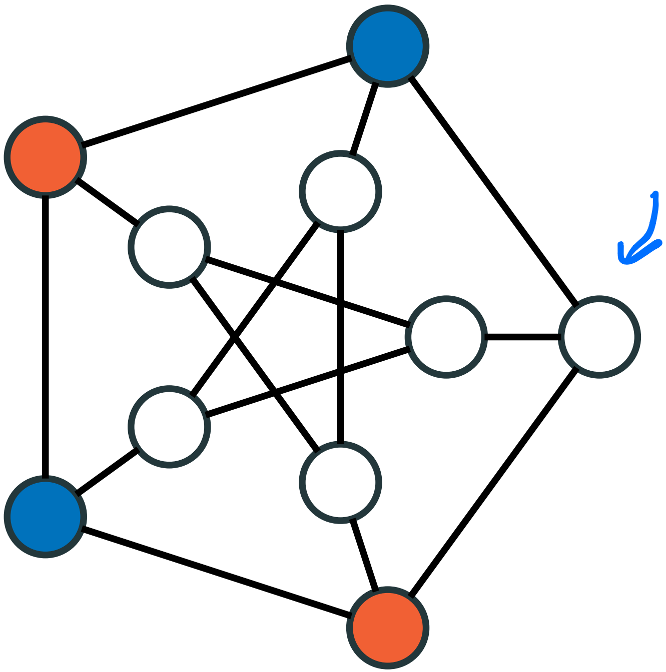

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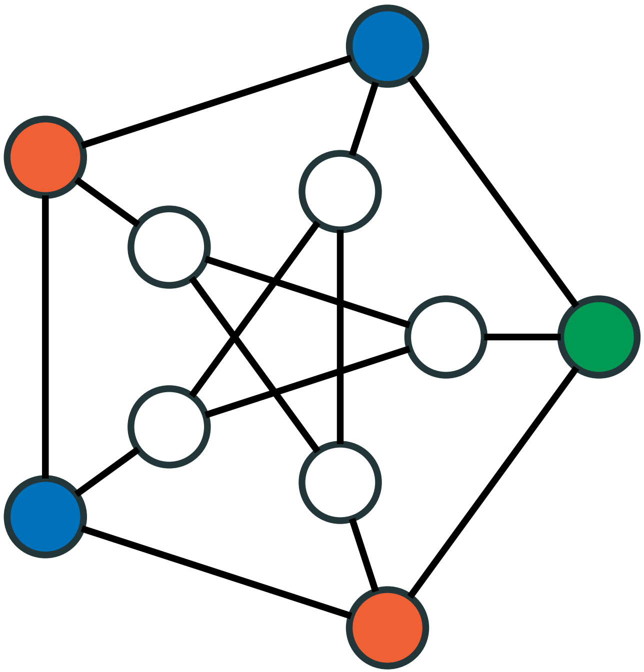

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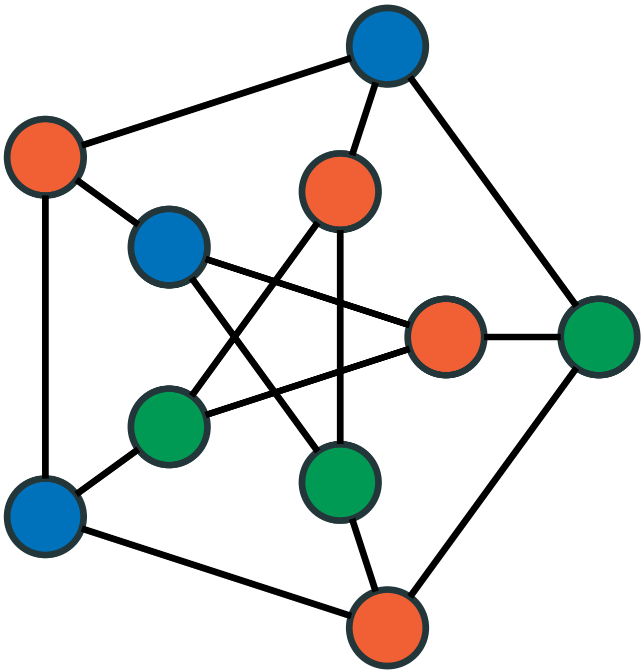
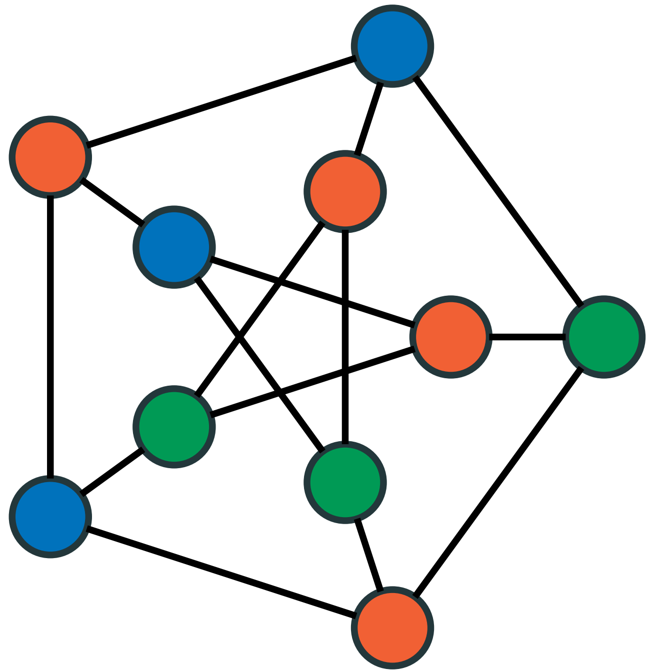


Chromatic Number

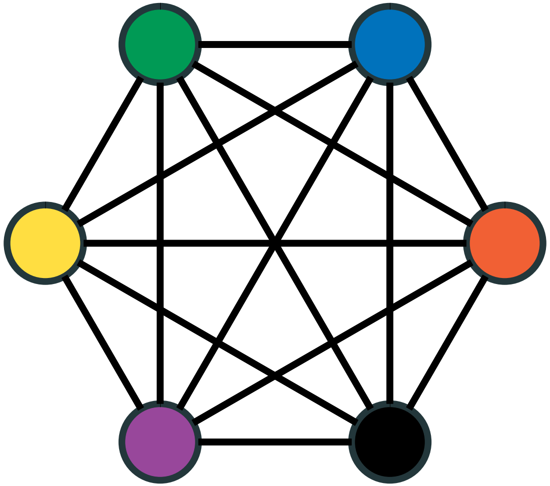
$\chi(G)=3$

Chromatic number is 3



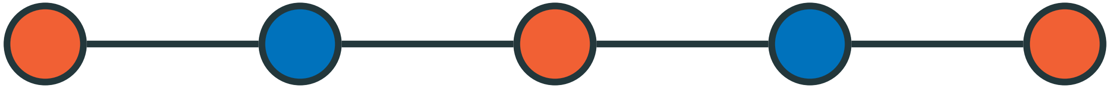
Complete Graphs

The chromatic number of K_n is n.



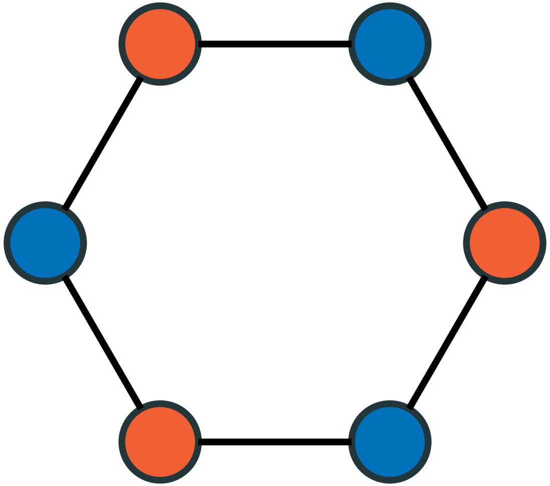
Path Graphs

For $n > 1$, the chromatic number of P_n is 2.



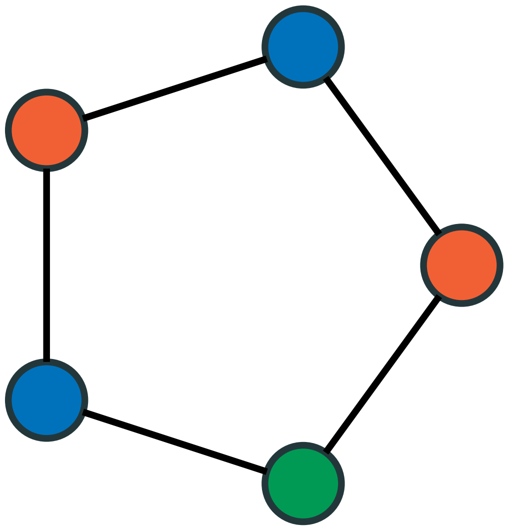
Cycle Graphs

For even n, the chromatic number of C_n is 2.



Cycle Graphs

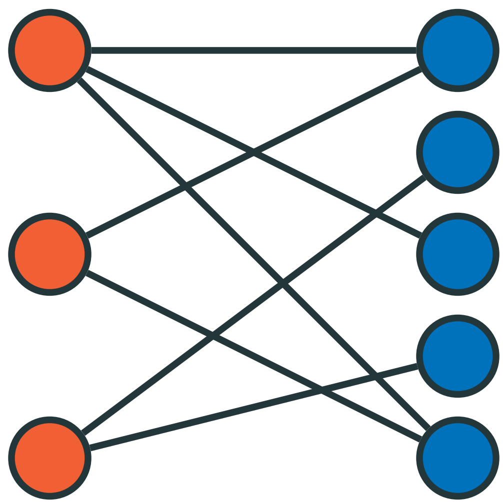
For odd $n > 2$, the chromatic number of C_n is 3.



Bipartite Graphs

– partition vertices into L and R, such that all edges connect $\ell \in L$ and $r \in R$

The chromatic number of a bipartite graph (with at least 1 edge) is 2.



Fact: given a 2-colorable graph, it's easy (linear time) to color it in 2 colors

Applications

Exam schedule

- Each student takes an exam in each of her courses
- All students in one course take the exam together
- One student cannot take two exams per day
- What is the minimum number of days needed for the exams?

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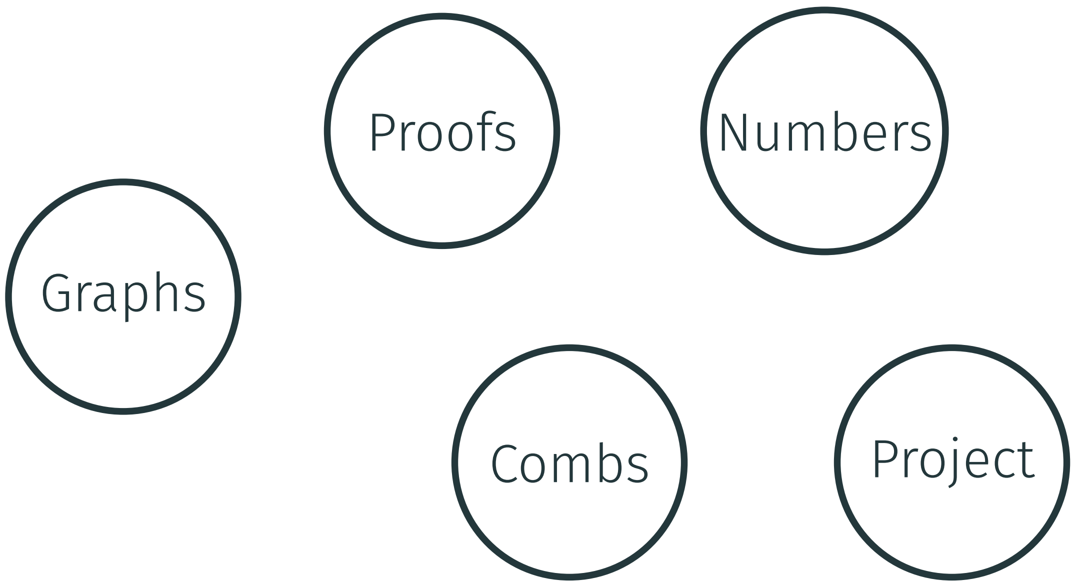

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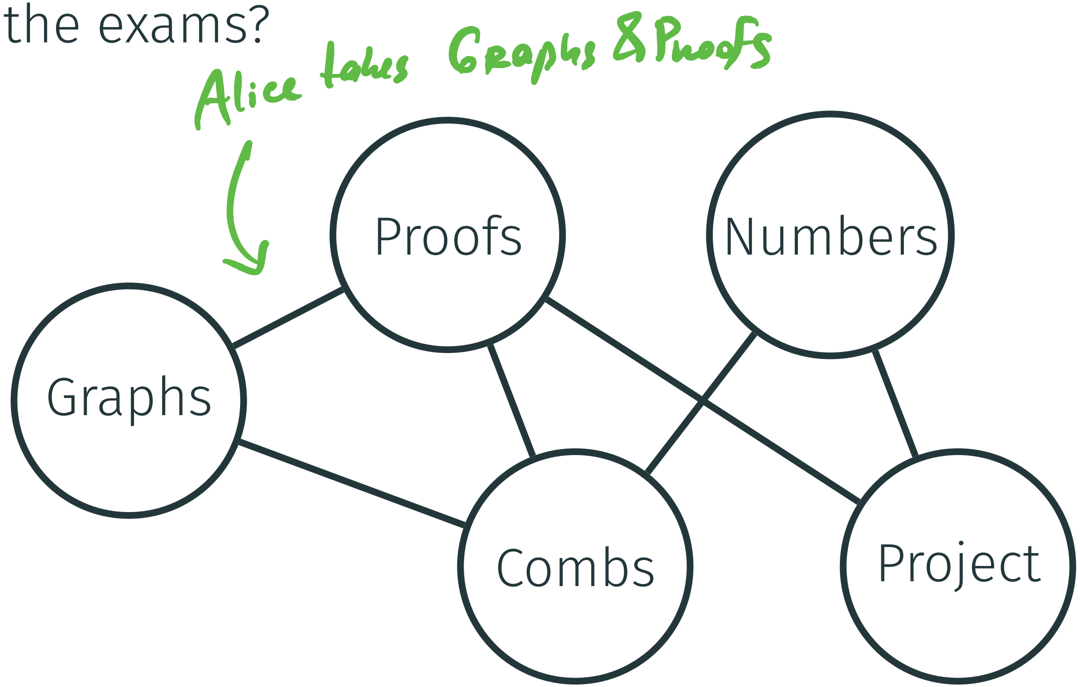

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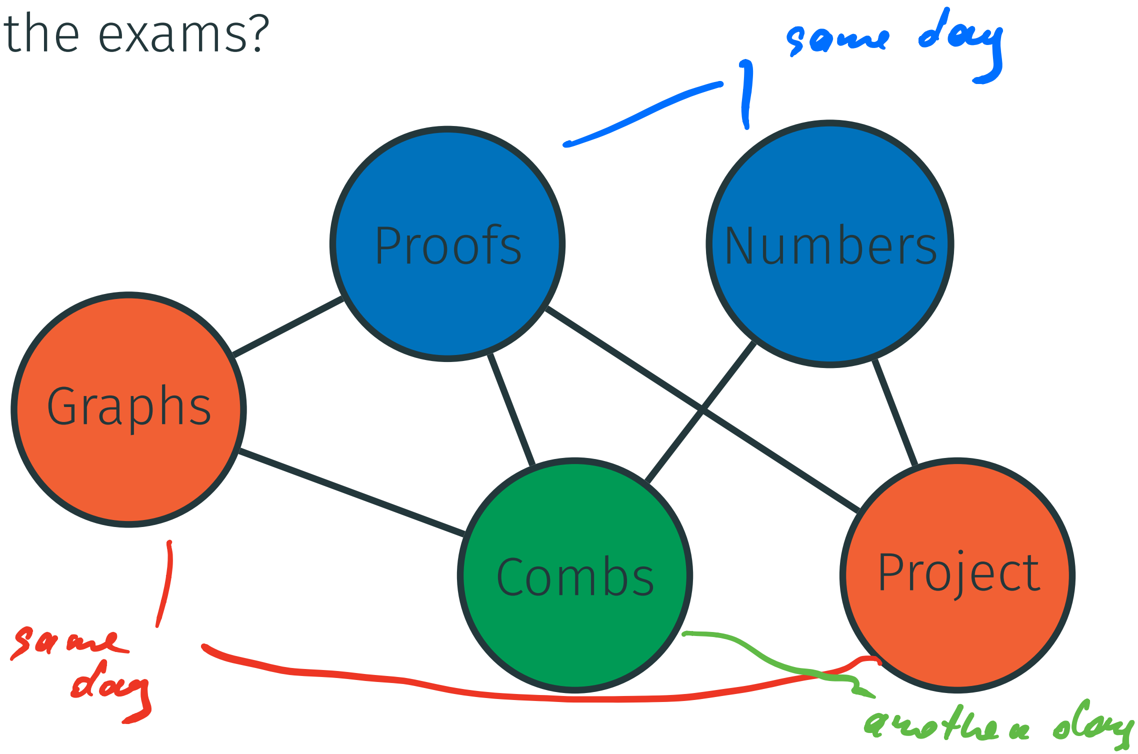
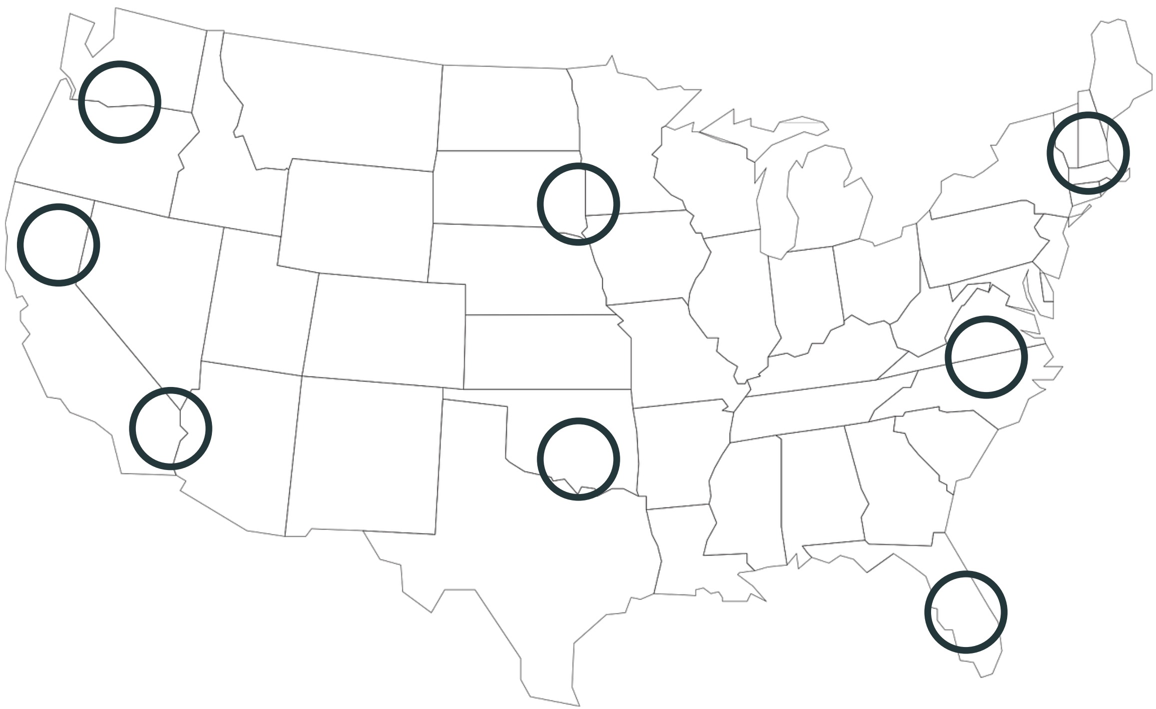


Bandwidth allocation

Different stations are allowed to use the same frequency if they are far apart. What is an optimal assignment of frequencies to stations?



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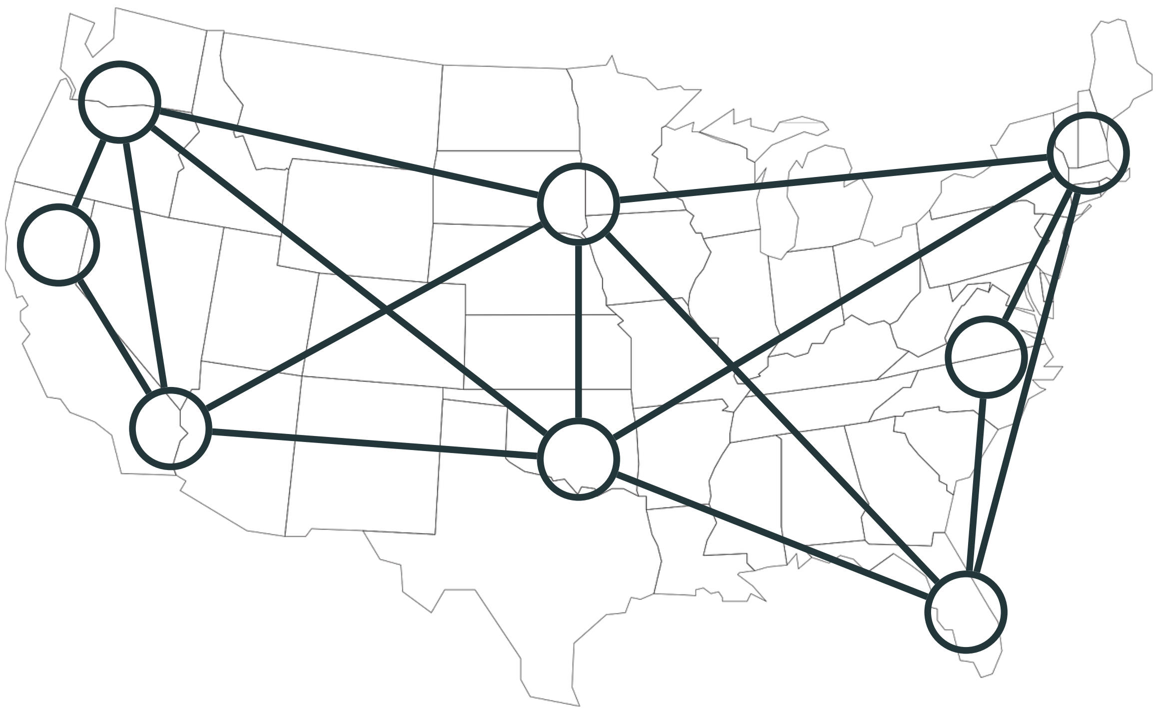

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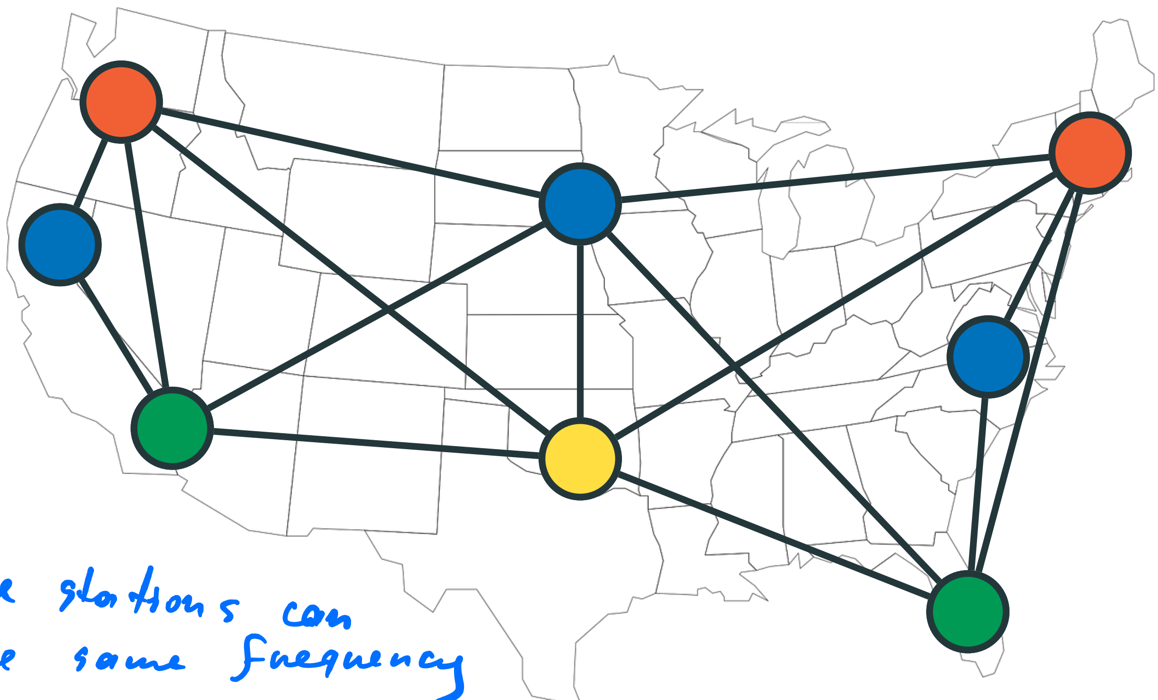

blue stations can use same frequency

Other Applications

- Scheduling Problems
- Register Allocation
- Sudoku puzzles
- Taxis scheduling
- ...

Exact Algorithm for Coloring

Dynamic Programming

- Given graph G on n vertices, find $\chi(G)$—minimum number of colors in a valid coloring of G

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- Given graph G on n vertices, find $\chi(G)$—minimum number of colors in a valid coloring of G
- Dynamic programming is one of the most powerful algorithmic techniques
- Rough idea: express a solution for a problem through solutions for smaller subproblems

Subproblems

- For a subset of vertices $S \subseteq \{1, \ldots, n\}$ compute $\chi(S)$—the minimum number of colors needed to color vertices S

Subproblems

$$\chi(G) = 1 + \chi(G \setminus U)$$

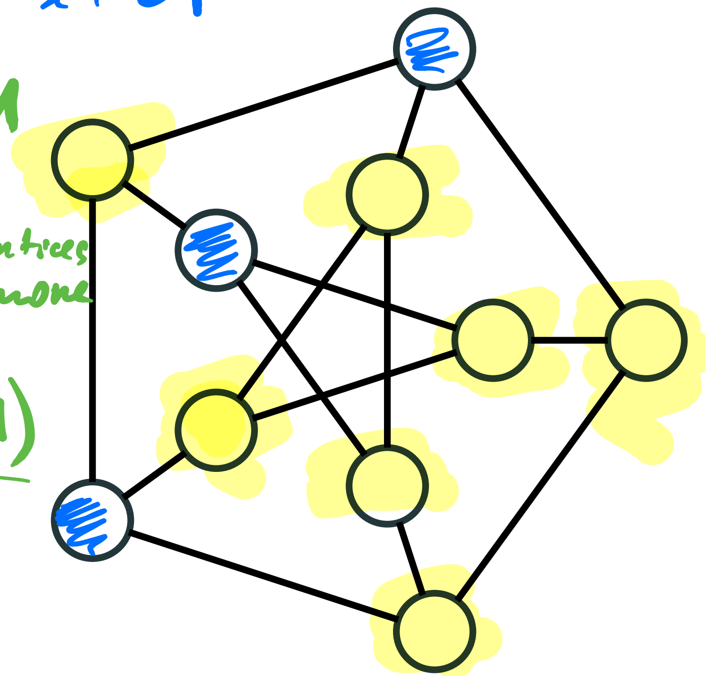
••• – set U

Try all subsets U

If $\exists$ edge between 2 vertices in U – ignore

Else

$1 + \chi(G \setminus U)$



Subproblems

- For a subset of vertices $S \subseteq \{1, \ldots, n\}$ compute $\chi(S)$—the minimum number of colors needed to color vertices S
- Consider S. For any subset $U \subseteq S$, if there are no edges between vertices from U, we can color them all in one color, and use $\chi(S \setminus U)$ to color the rest

$$\min_{U, \text{ s.t. no edges in } U} 1 + \chi(S \setminus U)$$

Subproblems

- For a subset of vertices $S \subseteq \{1, \ldots, n\}$ compute $\chi(S)$—the minimum number of colors needed to color vertices S
- Consider S. For any subset $U \subseteq S$, if there are no edges between vertices from U, we can color them all in one color, and use $\chi(S \setminus U)$ to color the rest

$$\chi(S) = \min_{U \text{ without edges}} 1 + \chi(S \setminus U)$$

Order of Subproblems

- Need to process all subsets $S \subseteq \{1, \ldots, n\}$ in order that guarantees that when computing the value of $\chi(S)$, the values of $\chi(S \setminus U)$ have already been computed

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- Need to process all subsets $S \subseteq \{1, \ldots, n\}$ in order that guarantees that when computing the value of $\chi(S)$, the values of $\chi(S \setminus U)$ have already been computed
- For example, we can process subsets in order of increasing size

Algorithm

$$\chi(\emptyset) = 0$$

Algorithm

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$s = |S|$

for s from 1 to n:

for all $S \subseteq \{1, \ldots, n\}$ of size s:

$\chi(S) = ?$

Algorithm

$\chi(\emptyset) = 0$

for s from 1 to n:

for all $S \subseteq \{1, \ldots, n\}$ of size s:

for all $U \subseteq S$, U without edges

$\chi(S) \leftarrow \min\{\chi(S), \chi(S \setminus U) + 1\}$

$\chi(\{1, \ldots, n\})$

Algorithm

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$\chi(S) \leftarrow \min\{\chi(S), \chi(S \setminus U) + 1\}$

return $\chi(\{1, \ldots, n\})$

Running Time

$\chi(\emptyset) = 0$

FOR s FROM 1 TO n: $\sum_{s=1}^{n}$

FOR ALL $S \subseteq \{1, \ldots, n\}$ OF SIZE s: $\binom{n}{s}$

FOR ALL $U \subseteq S$, U WITHOUT EDGES 2^s

$\chi(S) \leftarrow \min\{\chi(S), \chi(S \setminus U) + 1\}$

RETURN $\chi(\{1, \ldots, n\})$

$$\sum_{s=1}^{n} \binom{n}{s} \cdot 2^s = 3^n$$

# of choices S — # of choices U

Binomial thm: $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot x^i \cdot y^{n-i}$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$x = 2 \quad y = 1$

$3^n = (2+1)^n = \sum_{i=0}^{n} \binom{n}{i} 2^i \cdot 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i} 2^i$

Randomized Algorithm for 3-Coloring

Randomized Algorithm

NP-hard

- Given a 3-colorable graph, find a 3-coloring

Compare: 2-coloring ≡ Bipartite
in poly time

Recall: 2-SAT in poly time
3-SAT NP-hard

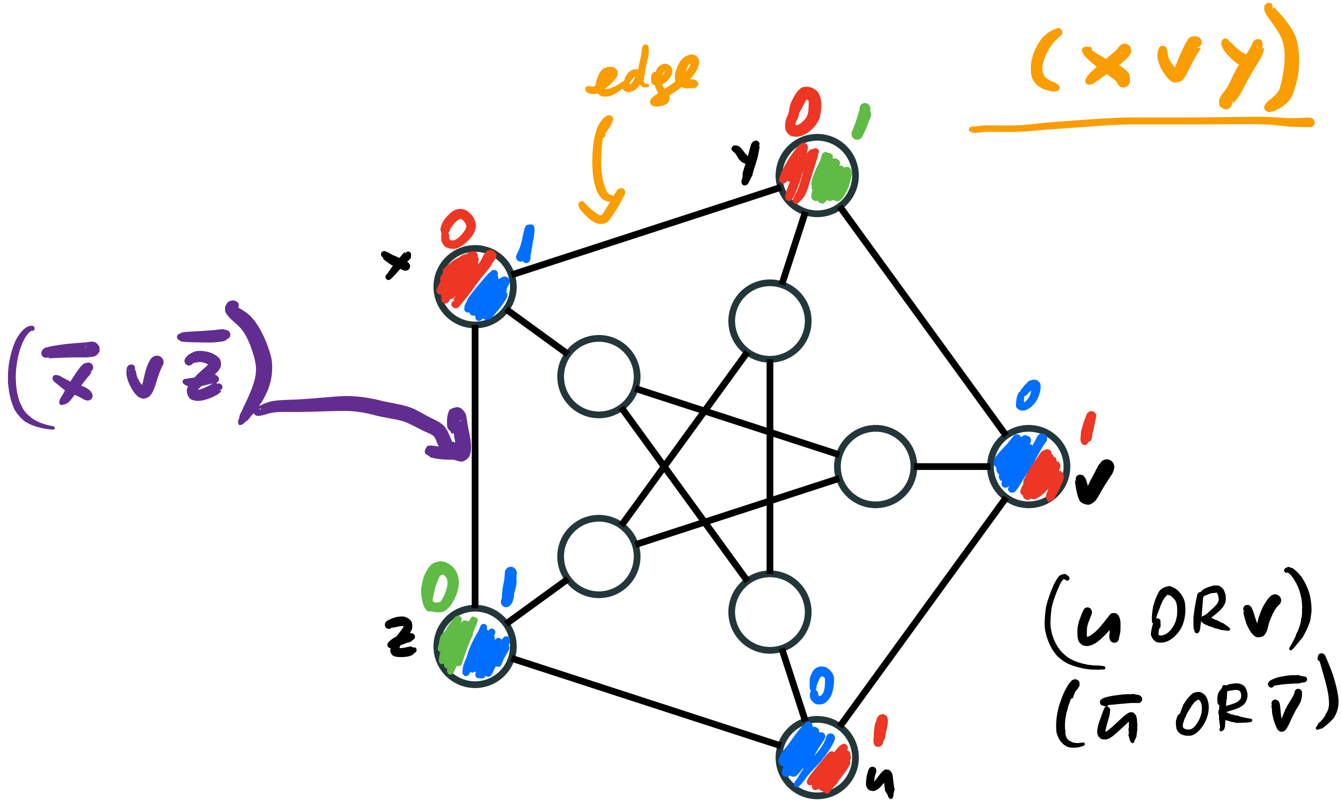
Randomized Algorithm

- Given a 3-colorable graph, find a 3-coloring
- This problem is **NP**-hard, we'll give an exponential-time algorithm

Randomized Algorithm

- Forbid one random color at each vertex

Randomized Algorithm



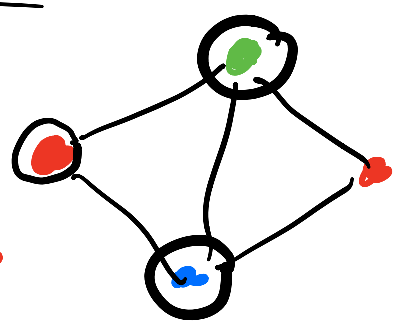
We get 2-SAT instance

2-SAT in linear time

Randomized Algorithm

- Forbid one random color at each vertex

- Solve 2-SAT in polynomial time



I forbid with 1/3

with probability 2/3, 1st vertex will be allowed use

In total, w.p. $\geq \left(\frac{2}{3}\right)^n$, every vertex is allowed to use its "correct" color

Randomized Algorithm

1.5^n times repeat:

- Forbid one random color at each vertex
- Solve 2-SAT in polynomial time

- Repeat the algorithm $(3/2)^n$ times

Randomized algorithm for 3-coloring that runs $(3/2)^n$ and succeeds with high probability

Approximate Algorithm for 3-Coloring

Approximate Coloring

- Given a 3-colorable graph, finding a 3-coloring is NP-hard

exp. time

Approximate Coloring

- Given a 3-colorable graph, finding a 3-coloring is **NP**-hard

- Given a 3-colorable graph, finding an n-coloring is trivial

Approximate Coloring

- Given a 3-colorable graph, finding a 3-coloring is NP-hard
- Given a 3-colorable graph, finding an n-coloring is trivial
- $3\sqrt{n}$ colors
- We'll see how to find an $O(\sqrt{n})$-coloring in polynomial time

Graphs of Bounded Degree

Greedy Coloring

A graph G where each vertex has degree $\leq \Delta$ can be colored with $\Delta + 1$ colors.

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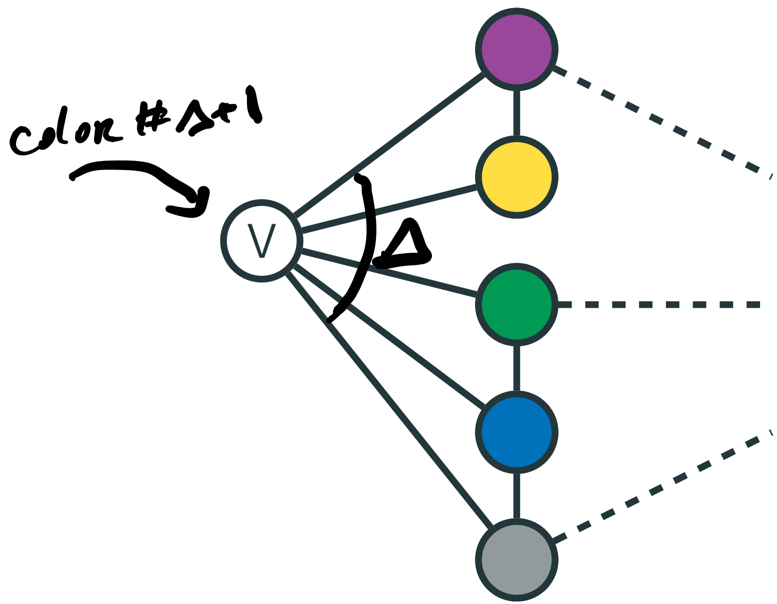
Greedy Coloring

A graph G where each vertex has degree Δ can be colored with $\Delta + 1$ colors.



Graphs of Bounded Degree

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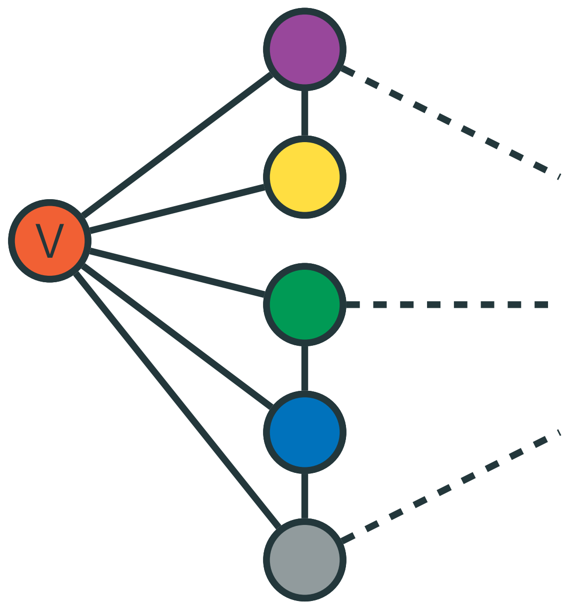
A graph G where each vertex has degree Δ can be colored with $\Delta + 1$ colors.



Approximate Algorithm For 3-coloring

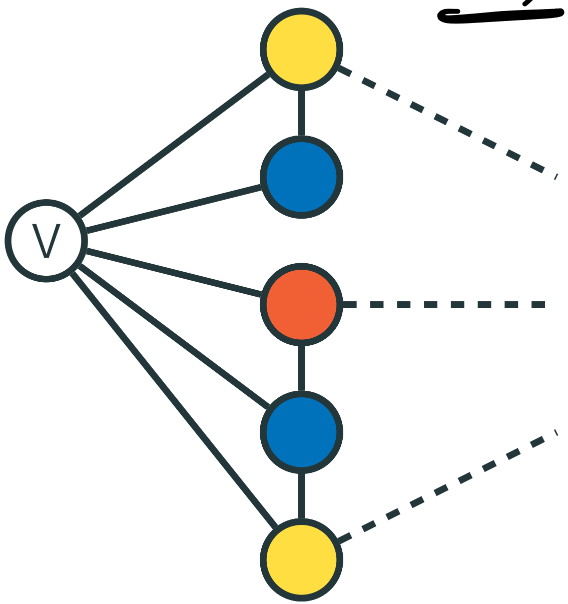
While there is vertex $v \in G$ of degree $\geq \sqrt{n}$:

Approximate Algorithm

While there is vertex $v \in G$ of degree $\geq \sqrt{n}$:

The set of its neighbors is 2-colorable $\equiv$ bipartite

<u>Proof:</u> Assume not.



V cannot use any of these 3 cols.

V requires 4th color.

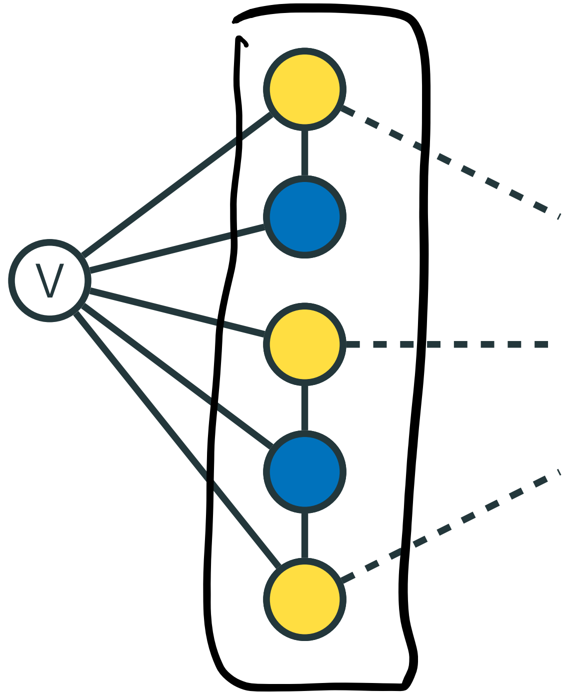
$\Rightarrow$ original graph is not 3-colorable.

Contradiction! $\square$

Approximate Algorithm

While there is vertex $v \in G$ of degree $\geq \sqrt{n}$:

Neighbors of v can be 2-colored, I can do this in linear time



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Color the neighbors of v in 2 new colors,

remove them from the graph

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While there is vertex $v \in G$ of degree $\geq \sqrt{n}$:
Color the neighbors of v in 2 new colors, remove them from the graph

$2\sqrt{n}$ colors

All remaining vertices have degree $< \sqrt{n}$. Color the rest of the graph using $\sqrt{n}$ new colors

$\sqrt{n}$ colors

Degree $\Delta \leq \sqrt{n} - 1$
Recall: can color $\Delta + 1 = \sqrt{n}$ colors.

used colors.
In each iteration of while loop, using 2 new colors.
How many iterations? n vertices in graph,
Removing $\geq \sqrt{n}$ of them $\Rightarrow$ # iterations $\leq \sqrt{n}$
After loop: using $\leq \sqrt{n}$ colors
$\leq 3\sqrt{n}$ colors