Gems of TCS

Linear Programming

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Linear Programming

- Optimization problems: among all solutions satisfying certain constrainst find optimal one

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- Find shortest cycle through all vertices   TSP

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- Find maximum vertex ~~color~~ covers

Linear Programming

- Optimization problems: among all solutions satisfying certain constrainst find optimal one
- Find shortest cycle through all vertices
- Find optimal coloring
- Find maximum vertex color
- Linear programming: class of optimization problems where constraints and optimization criterion are linear functions

Avoiding Scurvy

- Orange costs \$1,
  grapefruit costs \$1;
  we have budget of \$2/day

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  grapefruit has 150gm of vitamin C,
  maximize daily vitamin C intake.

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x oranges

y grapefruits

Constraints

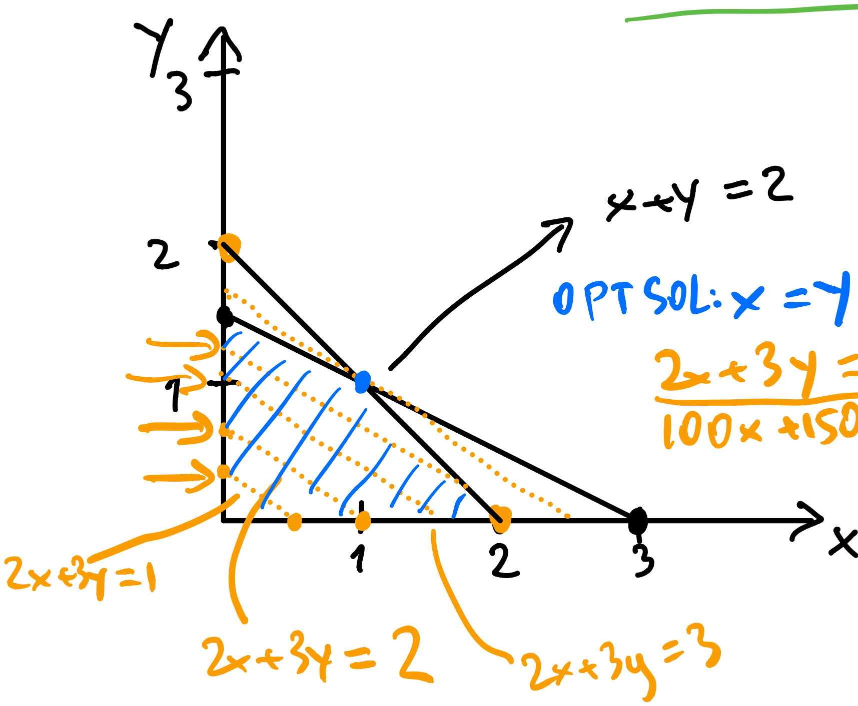
$$x \geq 0$$
$$y \geq 0$$
$$x + y \leq 2$$
$$100 \cdot x + 200 \cdot y \leq 300 \quad \Updownarrow \quad x + 2y \leq 3$$

$$2x + 3y \leq 5$$

$x^2 + xy \geq 7$ — non-linear

$$\text{Max } 100 \cdot x + 150y \quad \Updownarrow \quad \text{Max } 2x + 3y$$

$2xy - y^2$ — non-linear



$2x + 3y \geq 5$

OPT SOL: $x = y = 1$

$2x + 3y = 5$

$100x + 150y = 250$ gm

Profit Maximization

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w workers
m machines

2m workers operate machines

(w−2m) workers make chocolate without machines

How many chocolates/hour?

$$m \cdot 20 + (w-2m) \cdot 5$$
$$= 10m + 5w$$

Profit?

$$(10m + 5w) \cdot 10 - 40w$$
$$= 100m + 10w$$

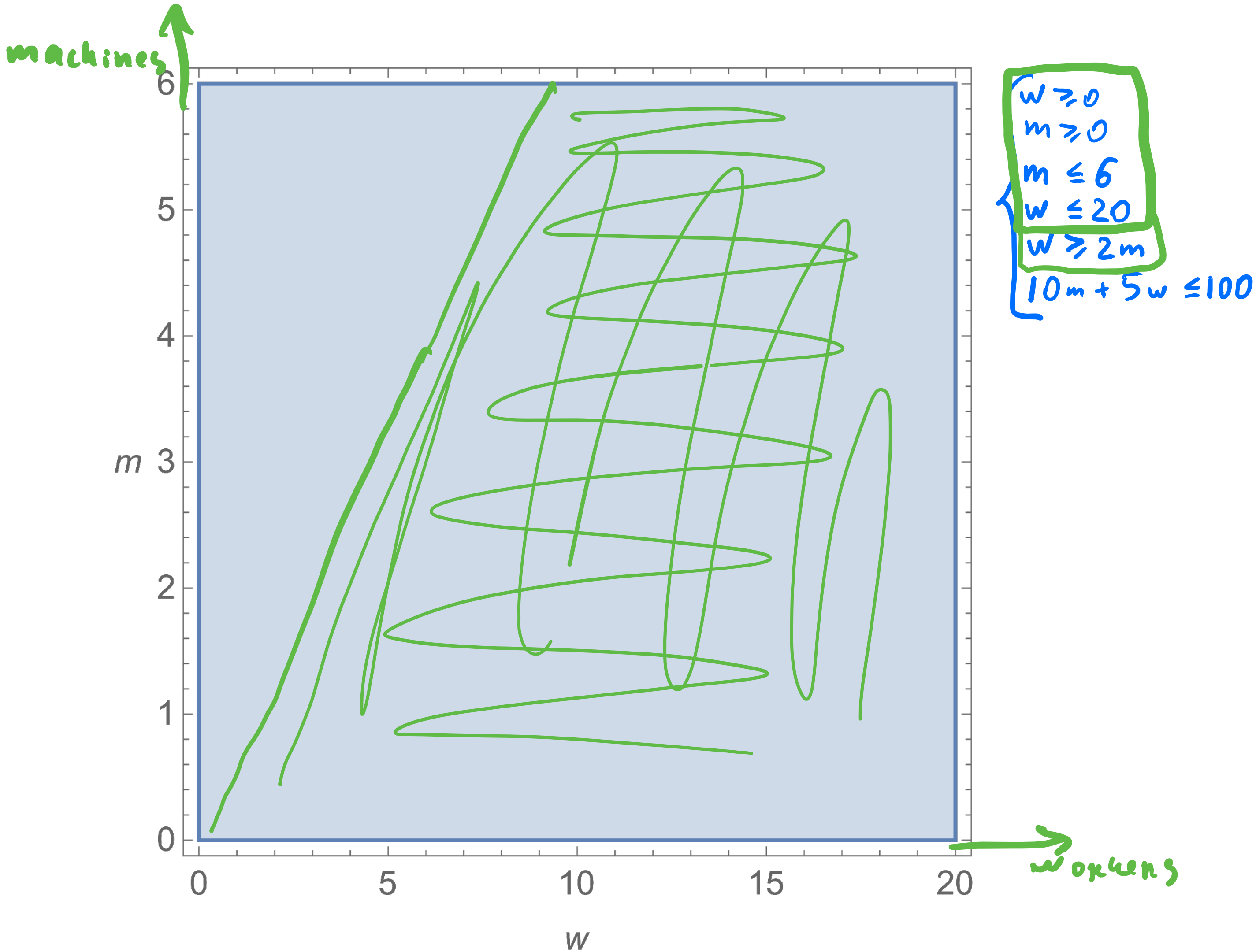
$$\begin{cases} w \geq 0 \\ m \geq 0 \\ m \leq 6 \\ w \leq 20 \\ w \geq 2m \\ 10m + 5w \leq 100 \end{cases}$$

linear

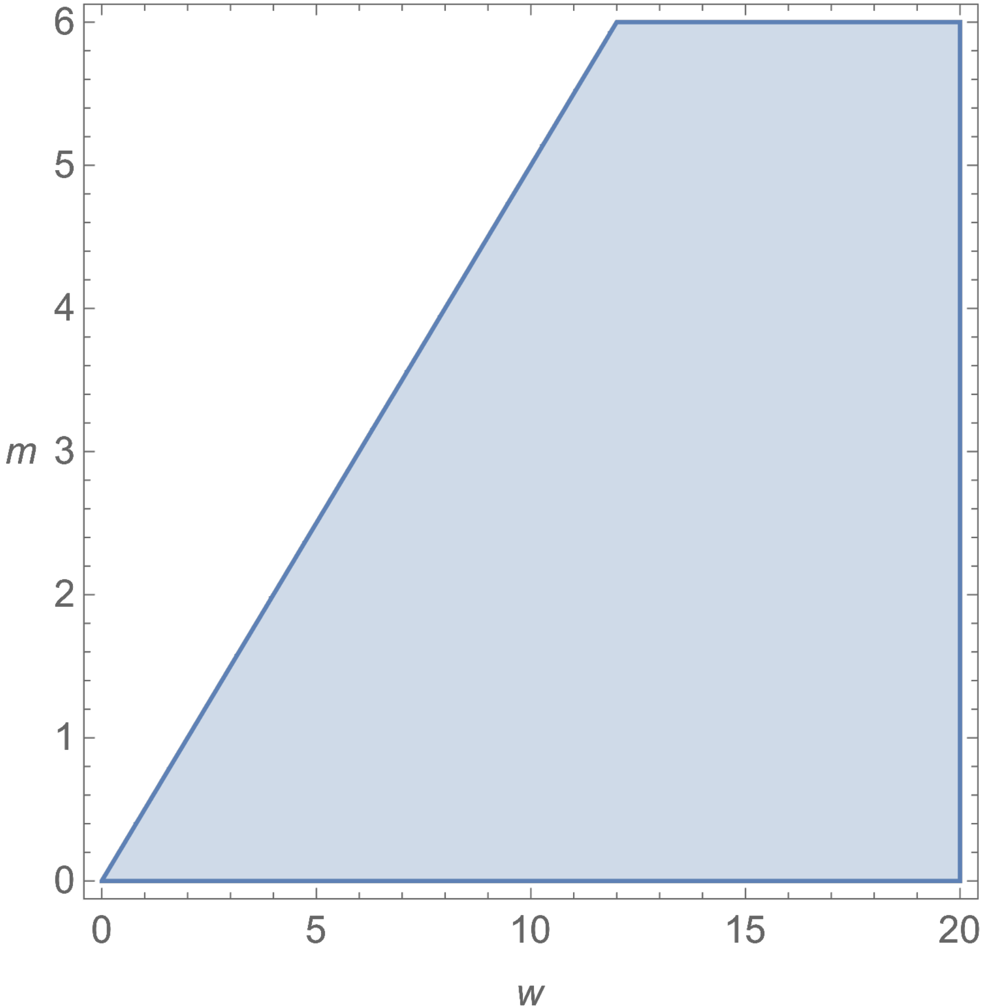
Max $100m + 10w$

linear

Workers and Machines

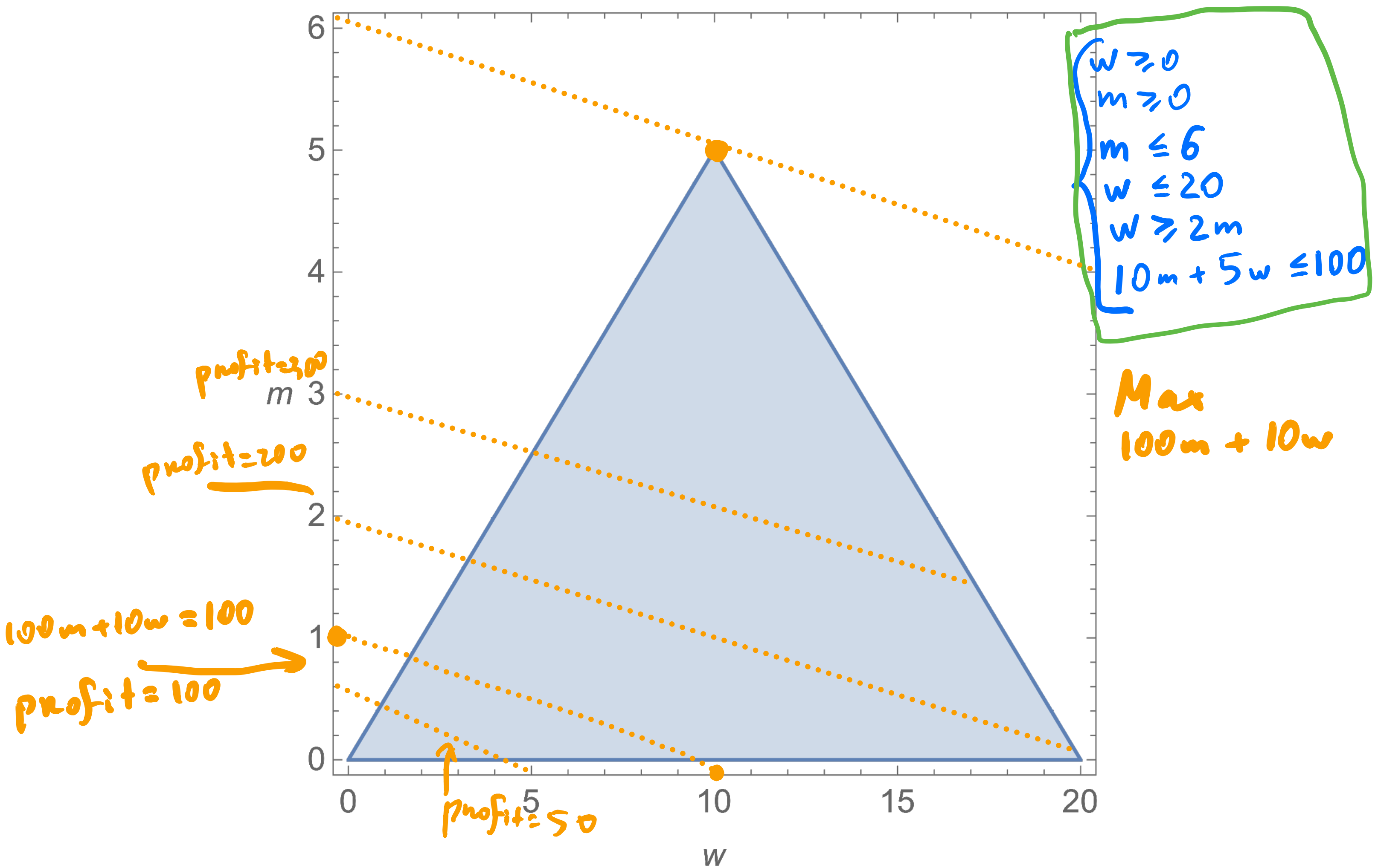


Two Workers Operate a Machine



$w \geq 0$
$m \geq 0$
$m \leq 6$
$w \leq 20$
$w \geq 2m$
$10m + 5w \leq 100$

Chocolate Demand



$w = 10 \quad m = 5$

$100m + 10w = 600$ ←

$w \leq 20$ *20
$w \geq 2m$
$10m + 5w \leq 100$ *6
←

$6(10m + 5w) + 20 \cdot 2m \leq$
$\leq 100 \cdot 6 + 20 \cdot w$

$60m + 30w + 40m \leq 600 + 20w$

$100m + 10w \leq 600$

Linear Classifier

Linear Classifier

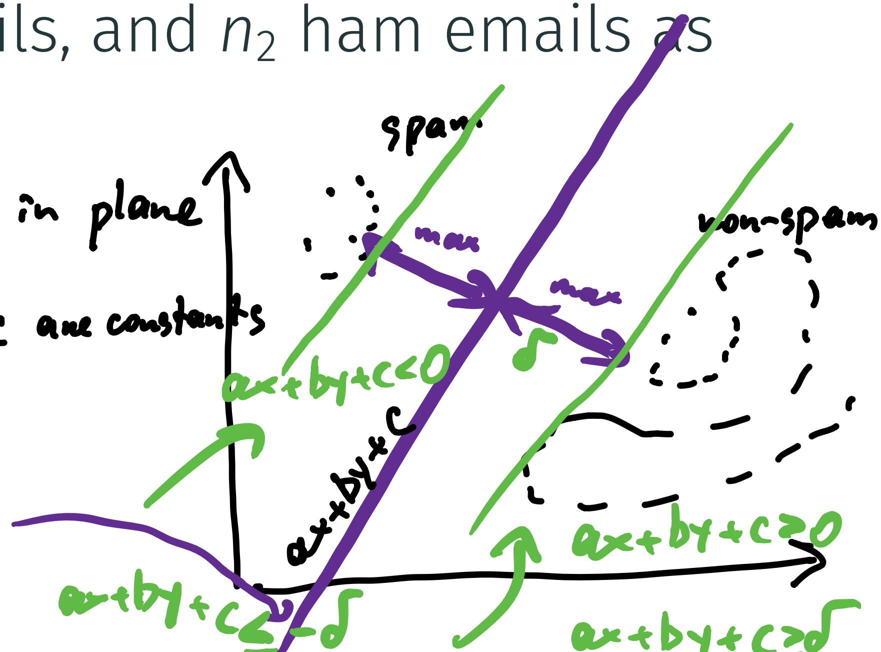
- Given n_1 spam emails, and n_2 ham emails as points in $\mathbb{R}^d$

each email – point in plane

$ax + by + c = 0$ — eq. of a line

$3x - 3.5y + 7 = 0$

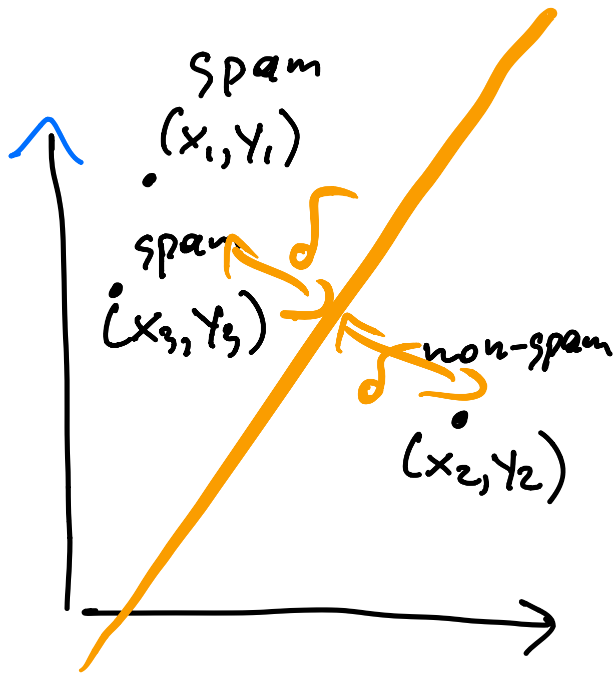
a, b, c are constants



Want to find $\underline{a, b, c}$

define line $ax + by + c = 0$

$$ax_1 + by_1 + c \leq -\delta$$
$$ax_2 + by_2 + c \geq \delta$$
$$ax_3 + by_3 + c \leq -\delta$$
$$\vdots$$



max δ

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emails in $\mathbb{R}^d$ where $d \approx 10^4$

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- Find a linear function $h(a_1, \ldots, a_d)$ s.t.
    - $h(a_1, \ldots, a_d) < 0$ for all spam emails
    - $h(a_1, \ldots, a_d) > 0$ for all ham emails

Linear Programming

Linear Programming

n variables $x_1, \ldots, x_n$

- Find real numbers $x_1, \ldots, x_n$ that satisfy linear constraints

$2x_1 + 3x_2 - 7x_7 \geq 10$

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \geq b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \geq b_m$$

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- So that linear objective is maximized

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$-x_1 + 7x_2 - 10x_{10}$

Equivalent Formulations

- Turn minimization problem into maximization problem:

replace

$$\begin{aligned} &\min && c_1x_1 + c_2x_2 + \ldots - c_nx_n \\ &\max && -c_1x_1 - c_2x_2 - \ldots - c_nx_n \end{aligned}$$

Equivalent Formulations

- Turn minimization problem into maximization problem:

$$\min \quad c_1x_1 + c_2x_2 + \ldots - c_nx_n$$
$$\max \; -c_1x_1 - c_2x_2 - \ldots - c_nx_n$$

- Turn $\leq$ into $\geq$:

replace

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$
$$-a_{11}x_1 - a_{12}x_2 - \ldots - a_{1n}x_n \geq -b_1$$

Equivalent Formulations

- Turn $=$ into $\geq$:

replace

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$$

$$-a_{11}x_1 - a_{12}x_2 - \ldots - a_{1n}x_n \geq -b_1$$

Max Flow — another example of LP

Matrix Formulation

Input is a matrix $A \in \mathbb{R}^{m \times n}$, and vectors $b \in \mathbb{R}^{m}$ and $c \in \mathbb{R}^{n}$

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$x \in \mathbb{R}^n$

$$\underline{Ax} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \vdots & \dots \\ \dots & \vdots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + & \dots & + a_{1n}x_n \\ \dots & \vdots & \dots \\ \dots & \vdots & \dots \\ a_{m1}x_1 + & \dots & + a_{mn}x_n \end{bmatrix} \geq \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

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$$Ax \geq b$$

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(i) $$Ax \geq b$$

(ii) maximize $cx = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = c_1x_1 + \dots + c_nx_n$

History of Linear Programming

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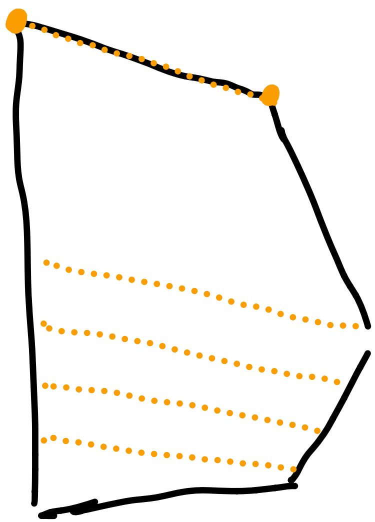
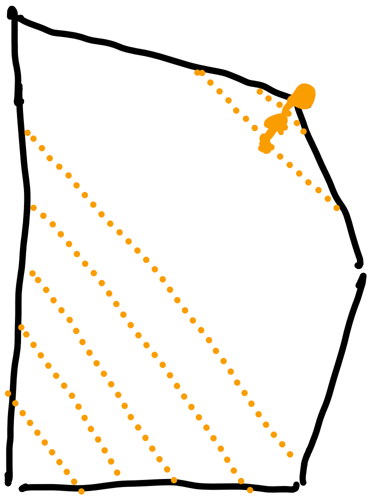
- Kantorovich and Koopmans won Nobel Prize in Economics in 1971
- Dantzig's algorithm is "One of top 10 algorithms of the 20th century"

Theorem

A linear function takes its maximum and minimum values on vertices





Simplex Method

Theorem

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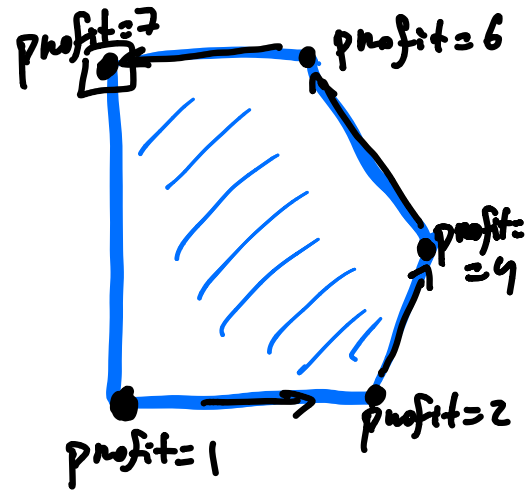
- Start at any vertex

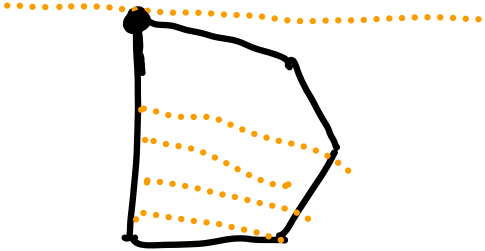


Simplex Method

Theorem

A linear function takes its maximum and minimum values on vertices

- Start at any vertex
- While there is an adjacent vertex with higher profit
  - Move to that vertex



Corner Cases

- No solutions

$$\begin{cases} x \geq 2 \\ y \geq 2 \\ x + y \leq 3 \end{cases} \qquad \emptyset$$

Corner Cases

- No solutions
- Unbounded profit

Algorithms for Simplex Method

– quite efficient in practice

- Simplex method

$0 \le x_1 \le 1$

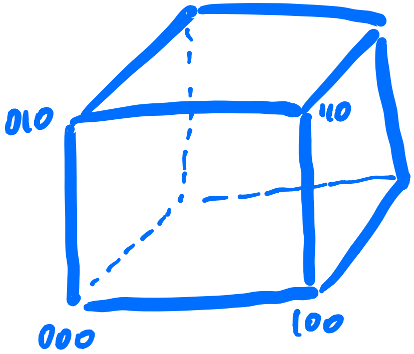
$0 \le x_2 \le 1$

$\vdots$

$0 \le x_n \le 1$

in n dimensions

2^n vertices



known examples where simplex method will have to check $\approx 2^n$ vertices

Algorithms for Simplex Method

- Simplex method
- Many professional packages that implement efficient algorithms for LP

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Khachiyan, 1979

Run-time n^7

non really practical

Algorithms for Simplex Method

- Simplex method ← efficiently in practice
- Many professional packages that implement efficient algorithms for LP
- Ellipsoid method
- Projective algorithm ← somewhat eff. in theory

  Karmarkar

  runtime $\approx n^{3.5}$

Algorithms for Simplex Method

- Simplex method
- Many professional packages that implement efficient algorithms for LP
- Ellipsoid method
- Projective algorithm
- Last week!

[JSWZ'21] solves LP in essentially n^2 time

Ellipsoid Method