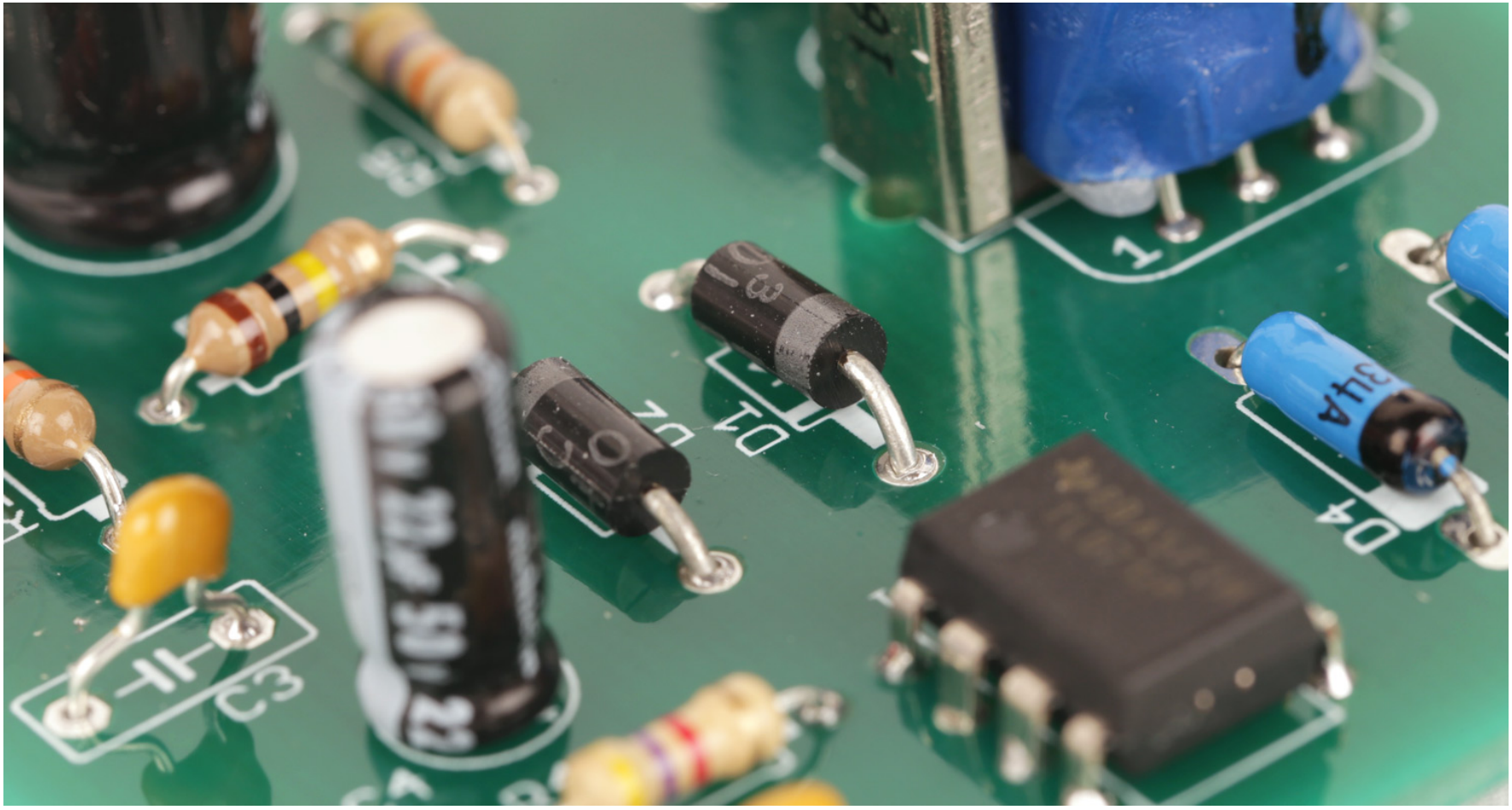
Gems of TCS

Circuit Complexity

Sasha Golovnev

October 23, 2023


C3
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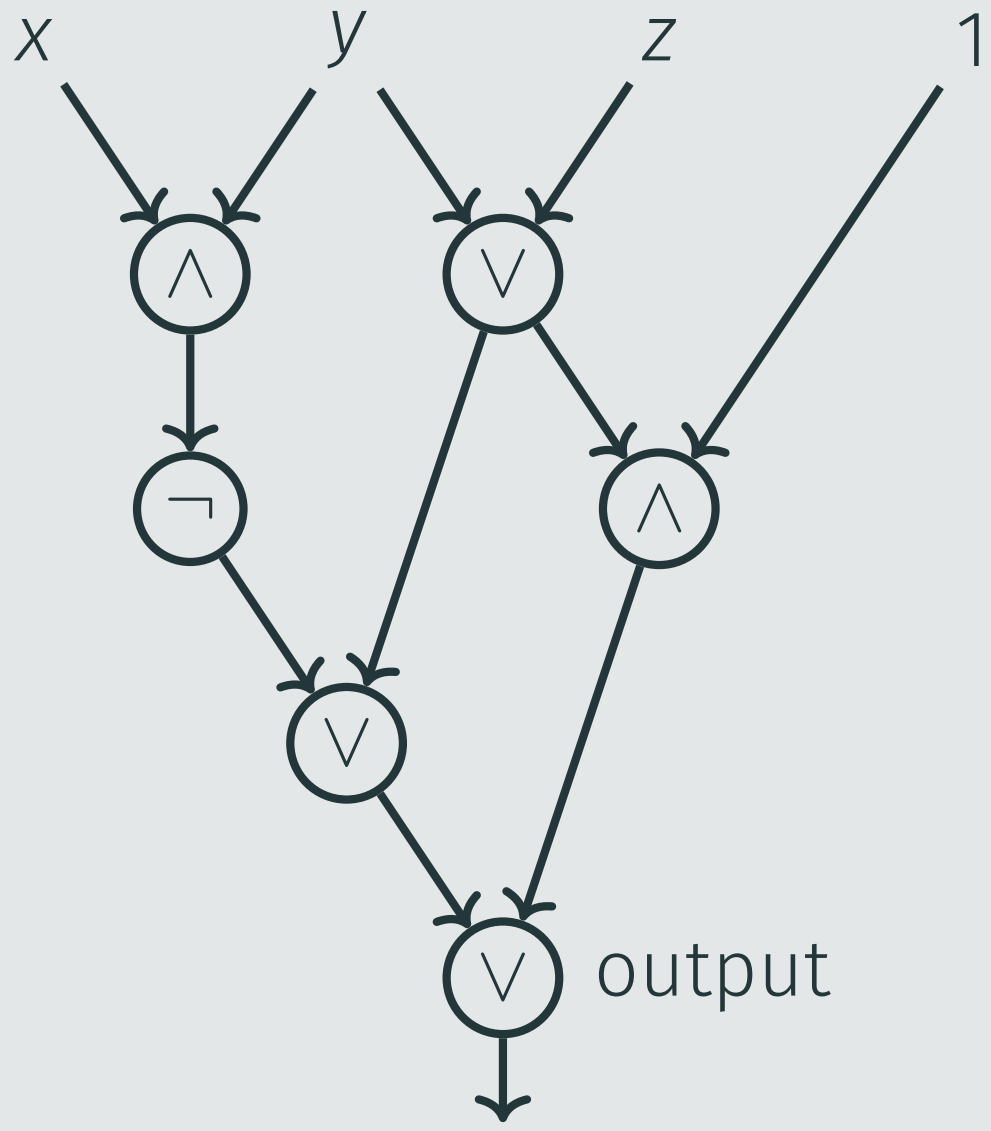
Circuit



Definition

A circuit is a directed acyclic graph of in-degree at most 2. Nodes of in-degree 0 are called inputs and are marked by Boolean variables and constants. Nodes of in-degree 1 and 2 are called gates: gates of in-degree 1 are labeled with NOT, gates of in-degree 2 are labeled with AND or OR. One of the sinks is marked as output.

Boolean Circuits

$$f\colon \{0,1\}^n \to \{0,1\}$$

$$\begin{aligned} g_1 &= \neg x_1 \\ g_2 &= x_2 \wedge x_3 \\ g_3 &= g_1 \vee g_2 \\ g_4 &= g_2 \vee 1 \\ g_5 &= g_3 \wedge g_4 \end{aligned}$$

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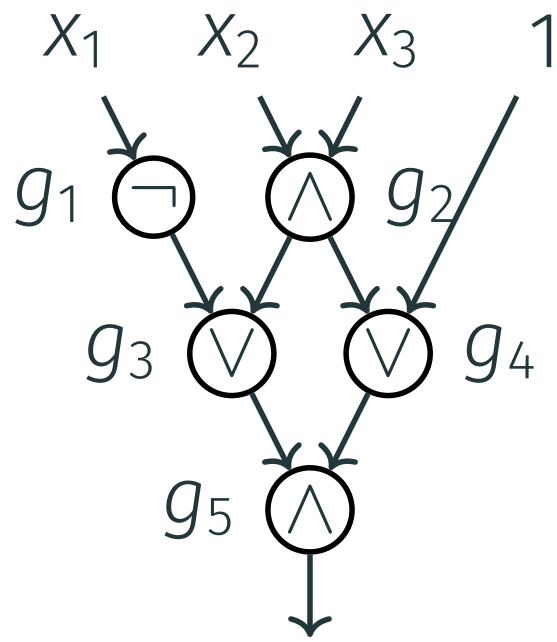

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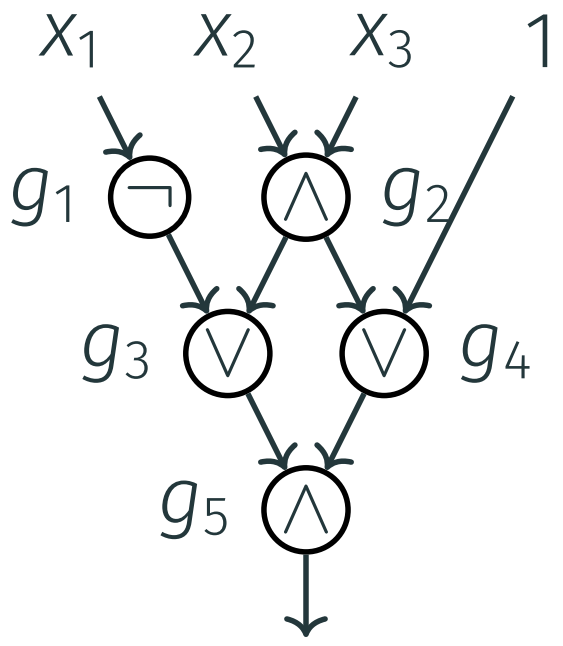

Inputs:

$x_1, \ldots, x_n, 0, 1$

Gates:

AND, OR, NOT

Fan-out:

unbounded

Depth:

unbounded

Exponential Bounds

Lower Bound [Sha1949]

Almost all functions of n variables have circuit size

$$\geq 2^n/n$$

Exponential Bounds

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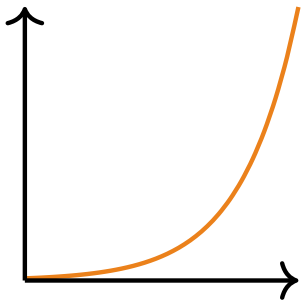
Upper Bound [Lup1958]

Any function can be computed by a circuit of size

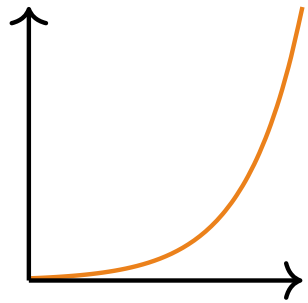
$$\leq 2^n/n$$

Explicit Bounds



Most functions have exponential circuit complexity

Explicit Bounds



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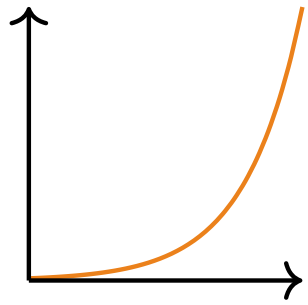
$P \neq NP$

We want to prove superpolynomial lower bounds

Explicit Bounds



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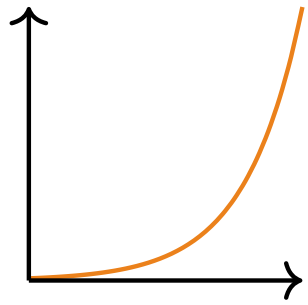

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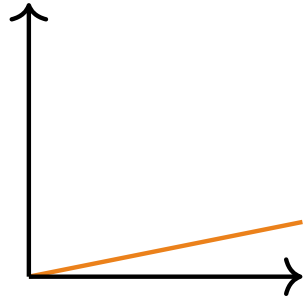
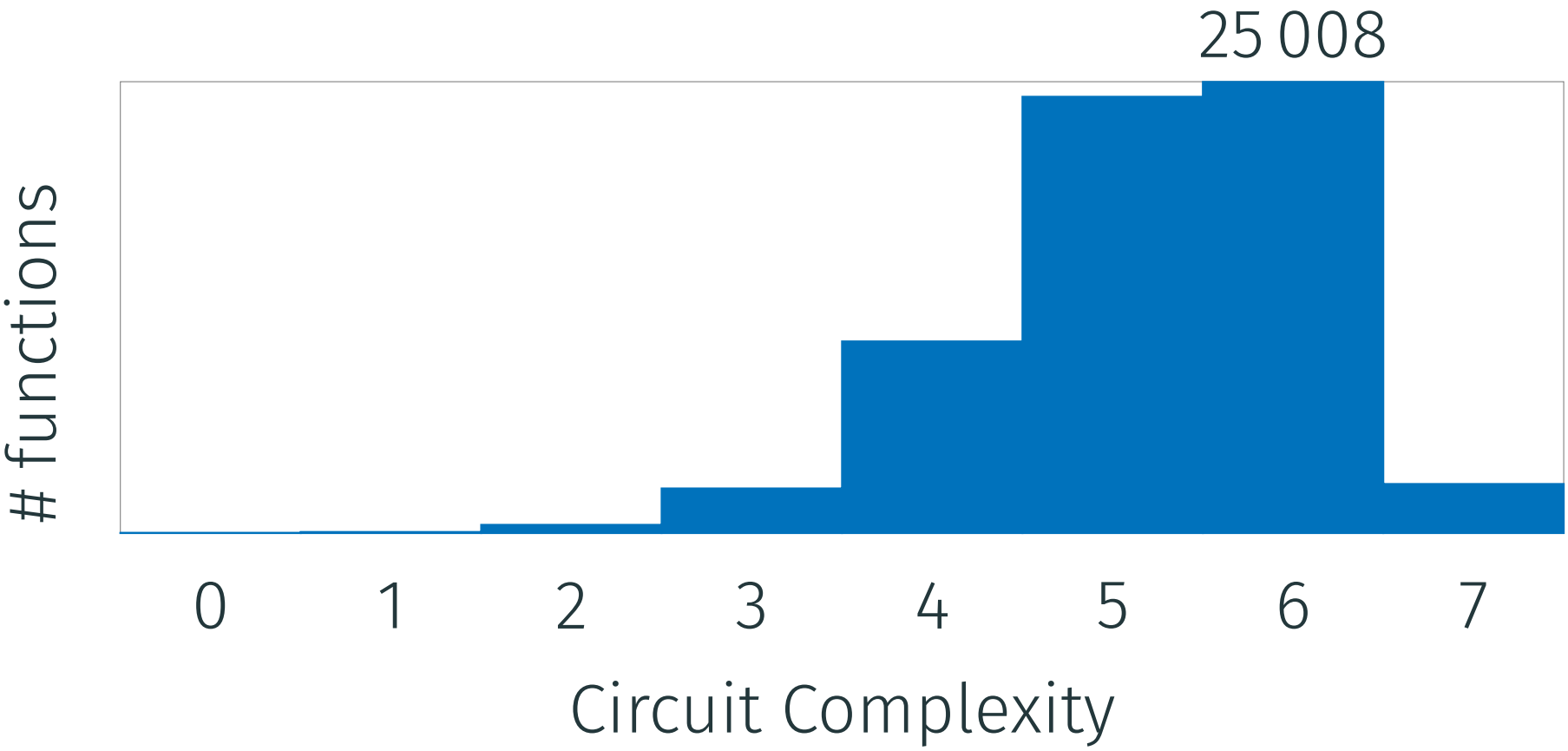


We can prove only $\approx 5n$ lower bounds

Circuit Complexity: $n = 4$



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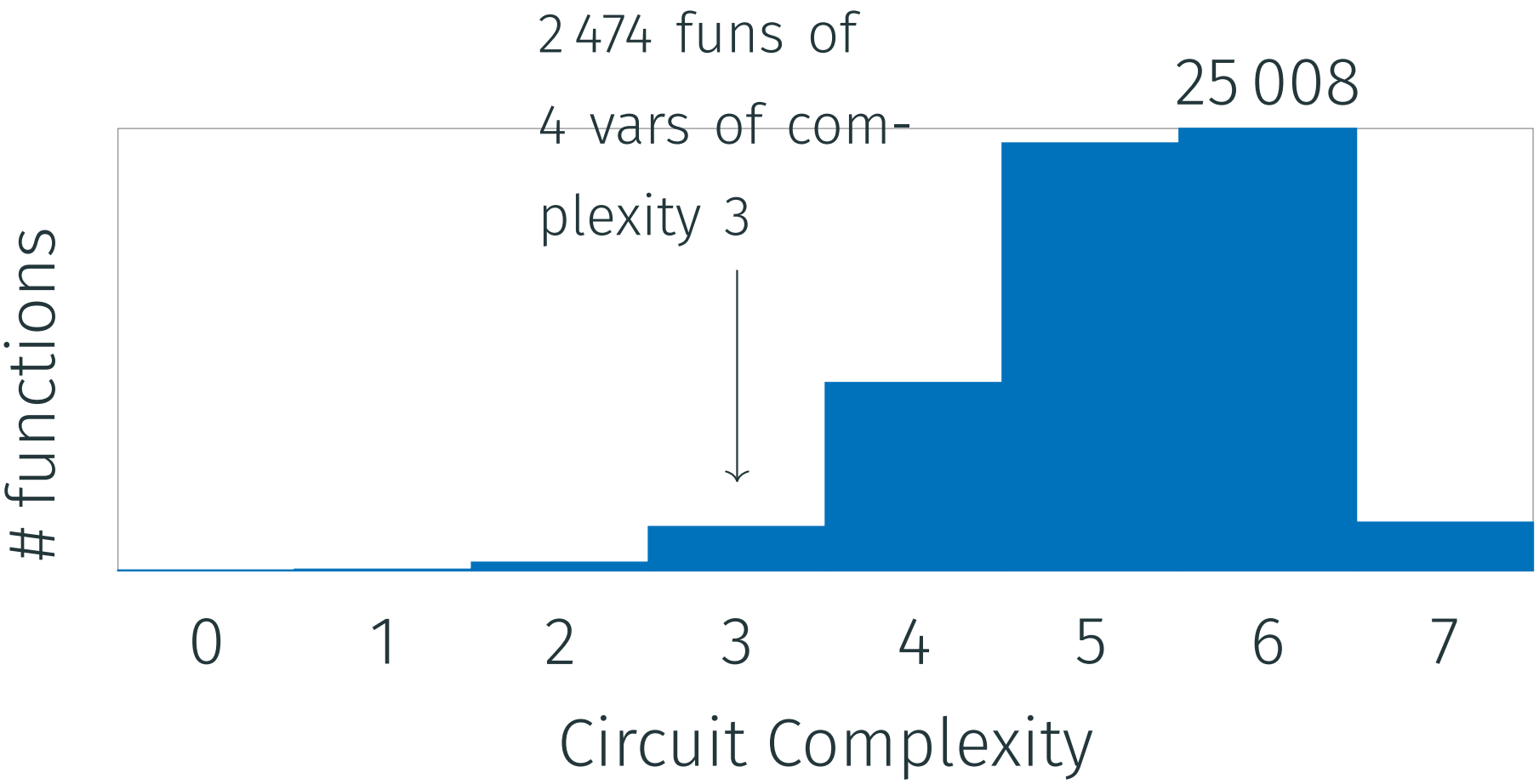
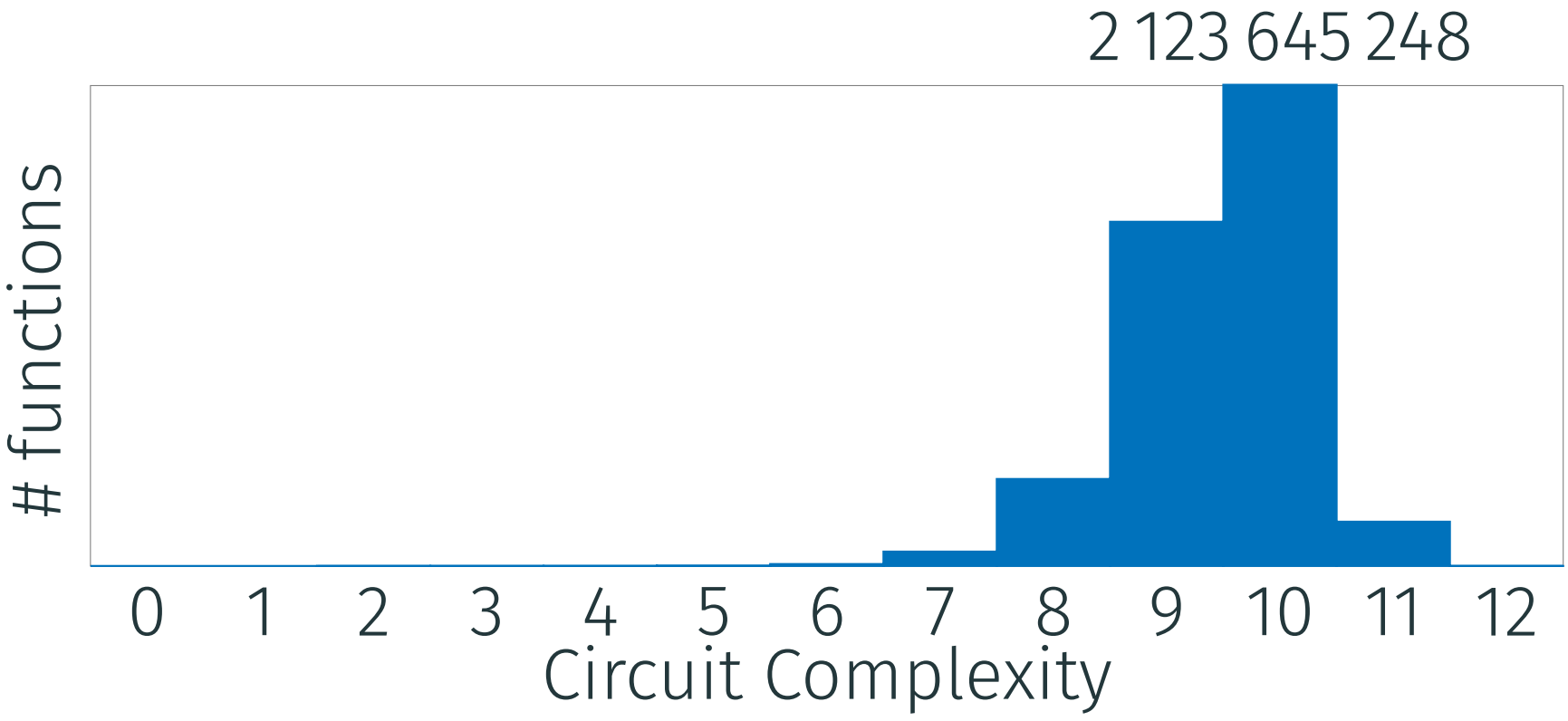
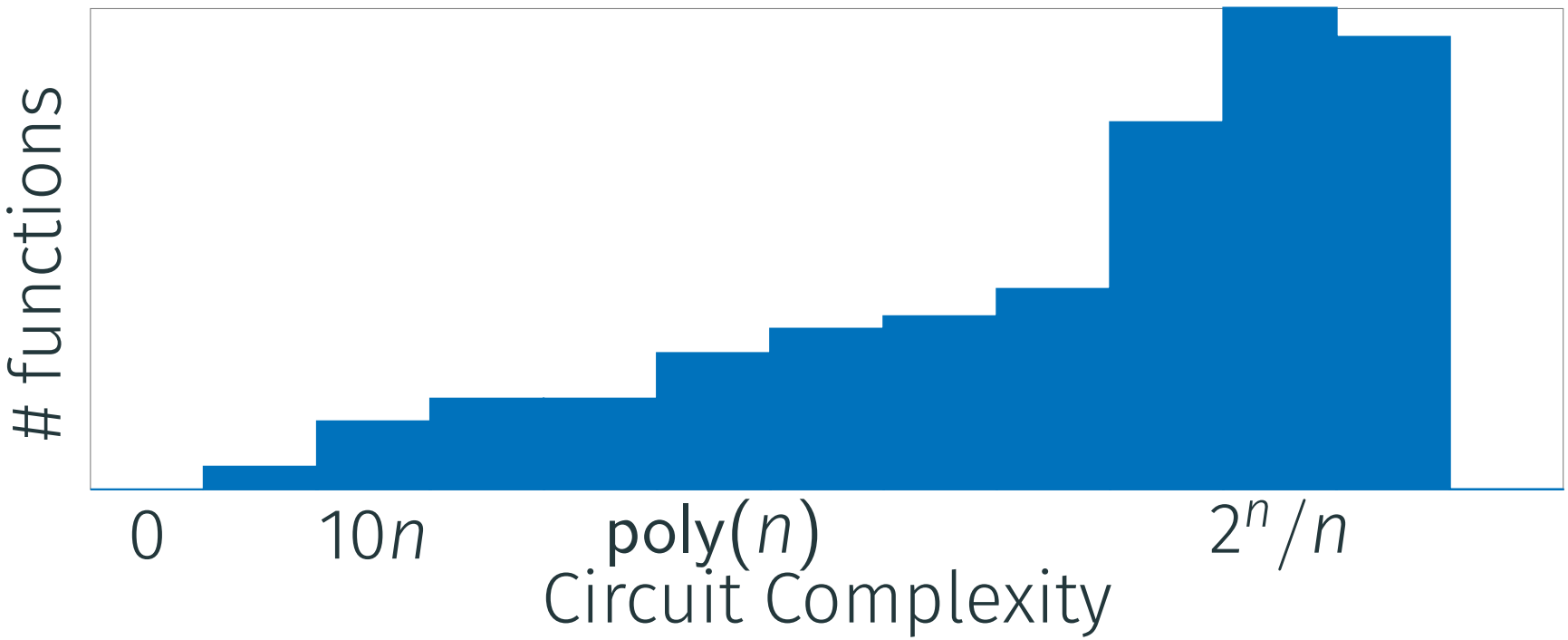


Circuit Complexity: $n = 5$



Circuit Complexity: General n



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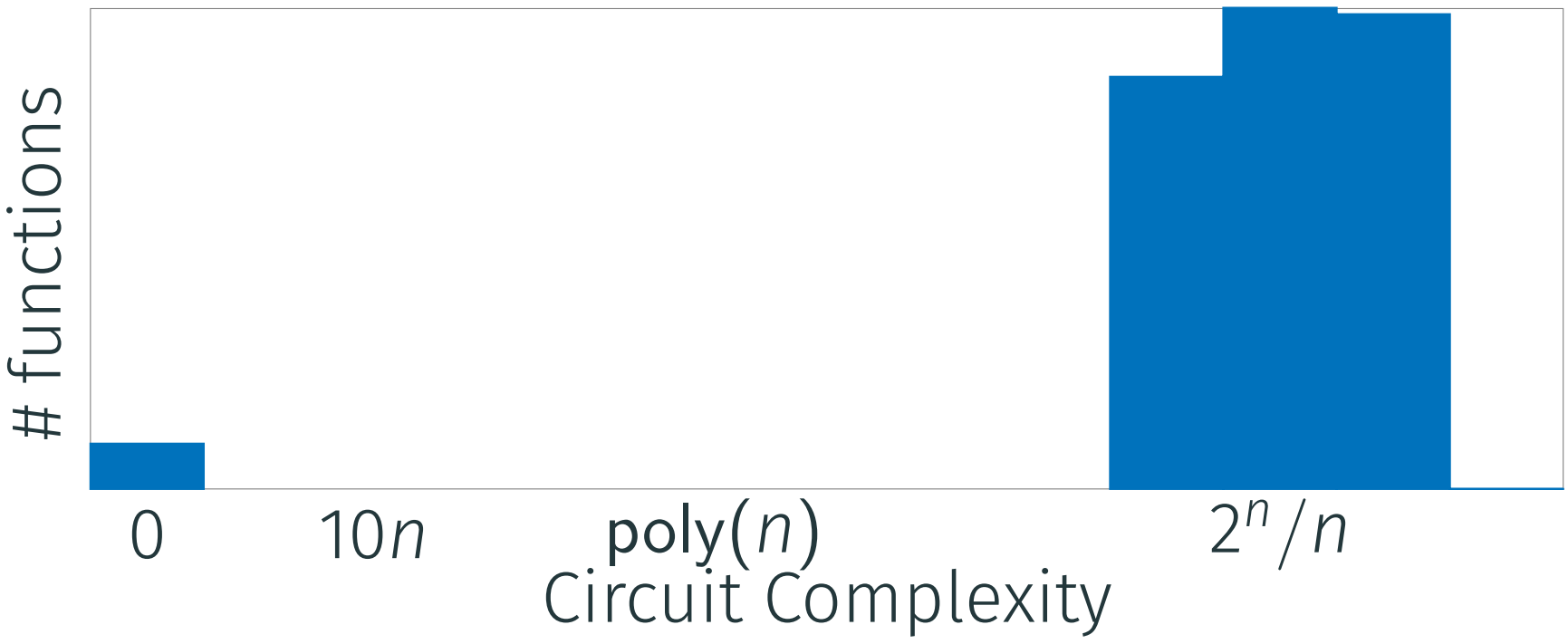

Hierarchy Theorem

Theorem

For any $T \leq 2^n/n$, there is a function $f\colon \{0,1\}^n \to \{0,1\}$ s.t.

$$\mathsf{Size}(f) = T \pm n\,.$$

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$h\colon \{0,1\}^n \to \{0,1\}$

$y_1, \ldots, y_k \in \{0,1\}^n$

$h(y_i) = 1$

Hybrid Method

$g_0(x) = 1$ never

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$$\text{Size}(g_{i+1}) \leq \text{Size}(g_i) + 2n$$

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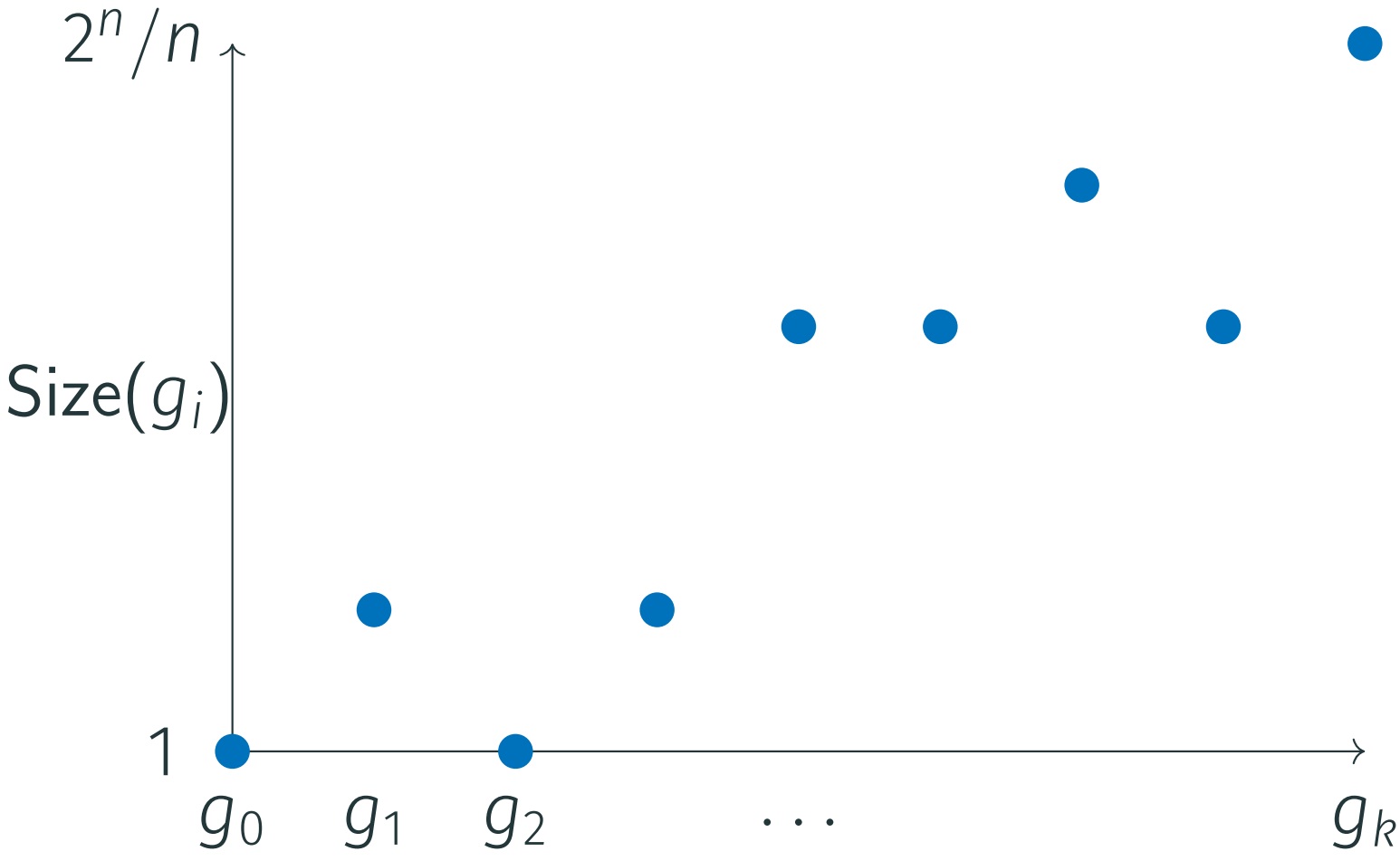

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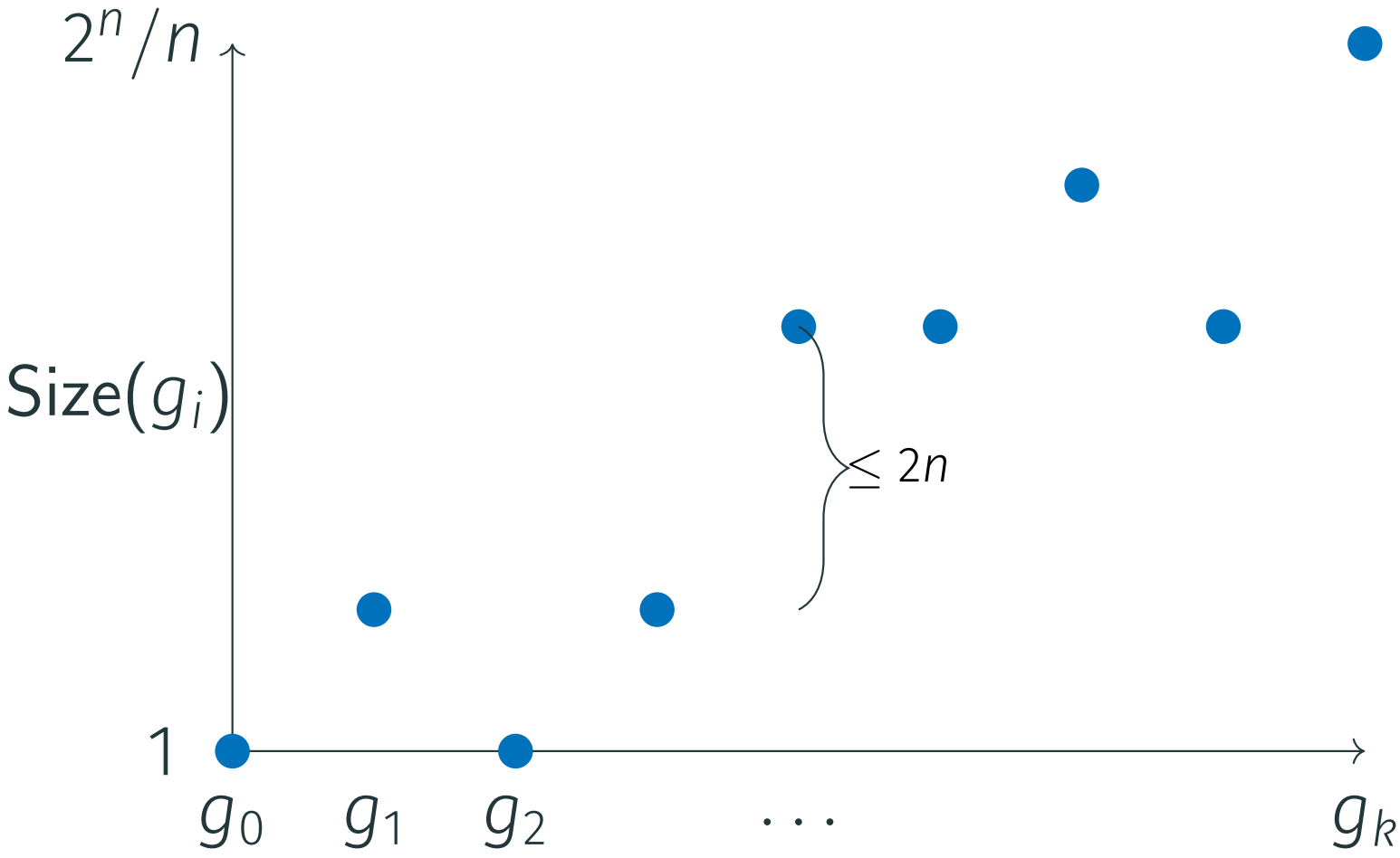

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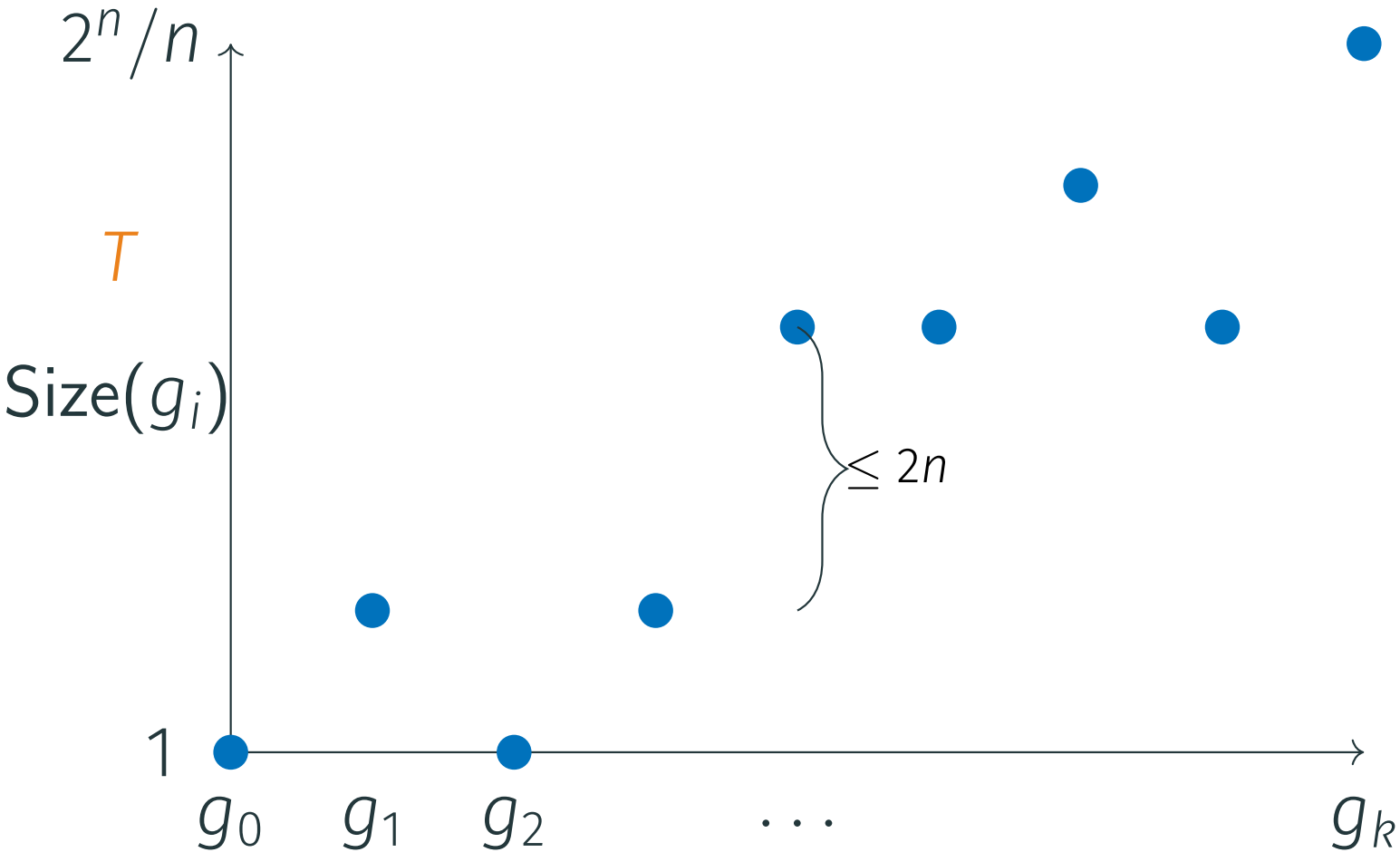

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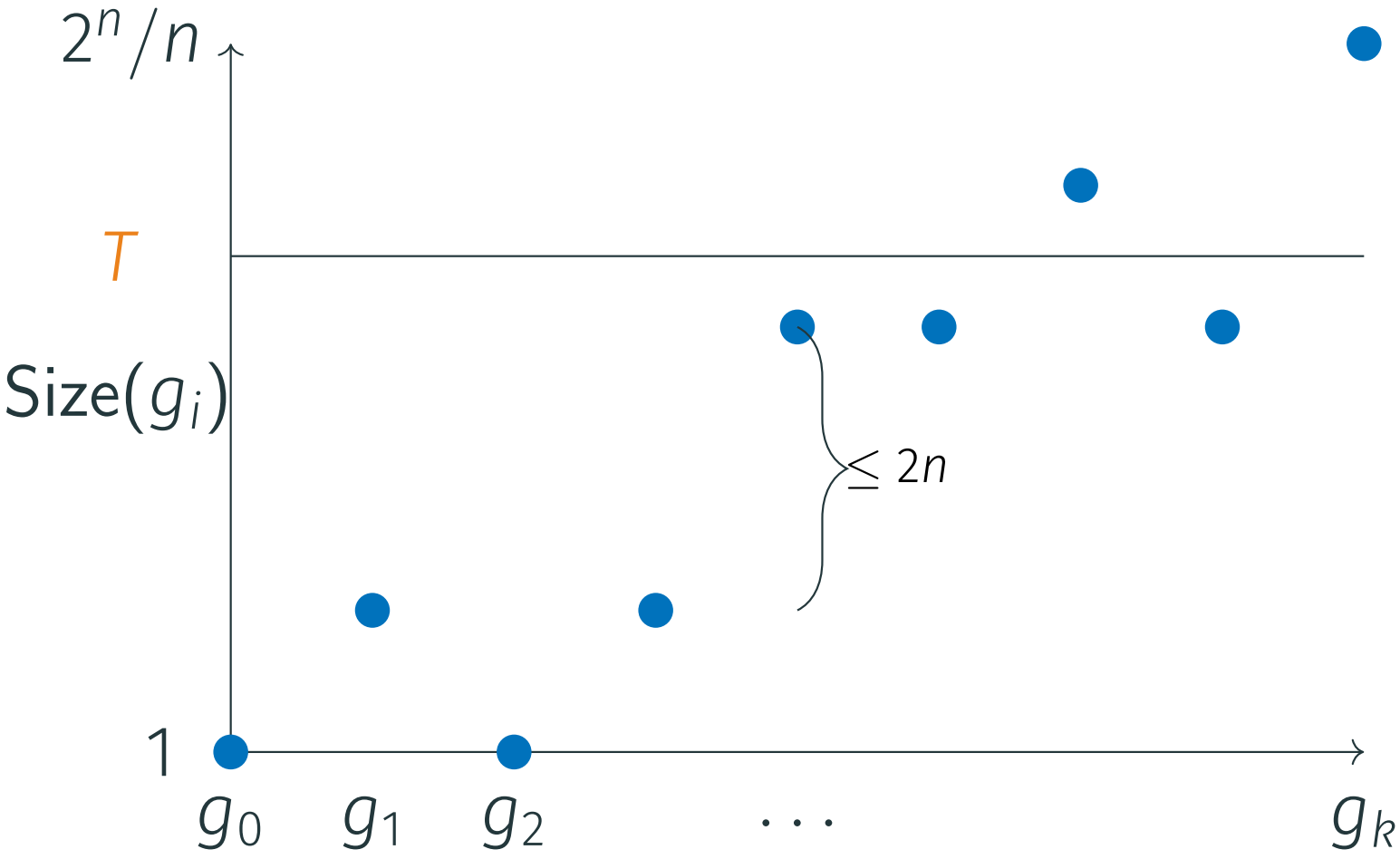

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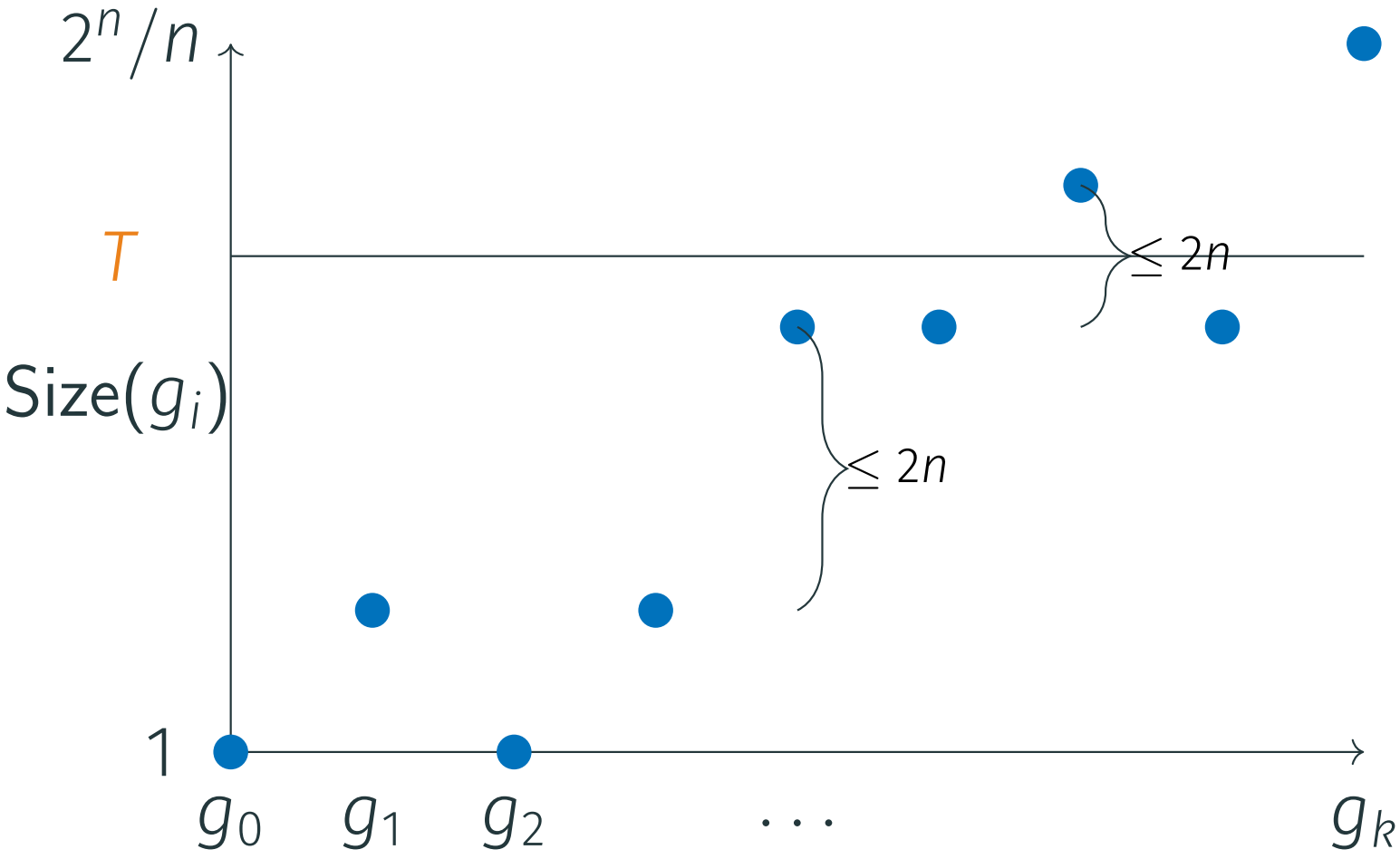

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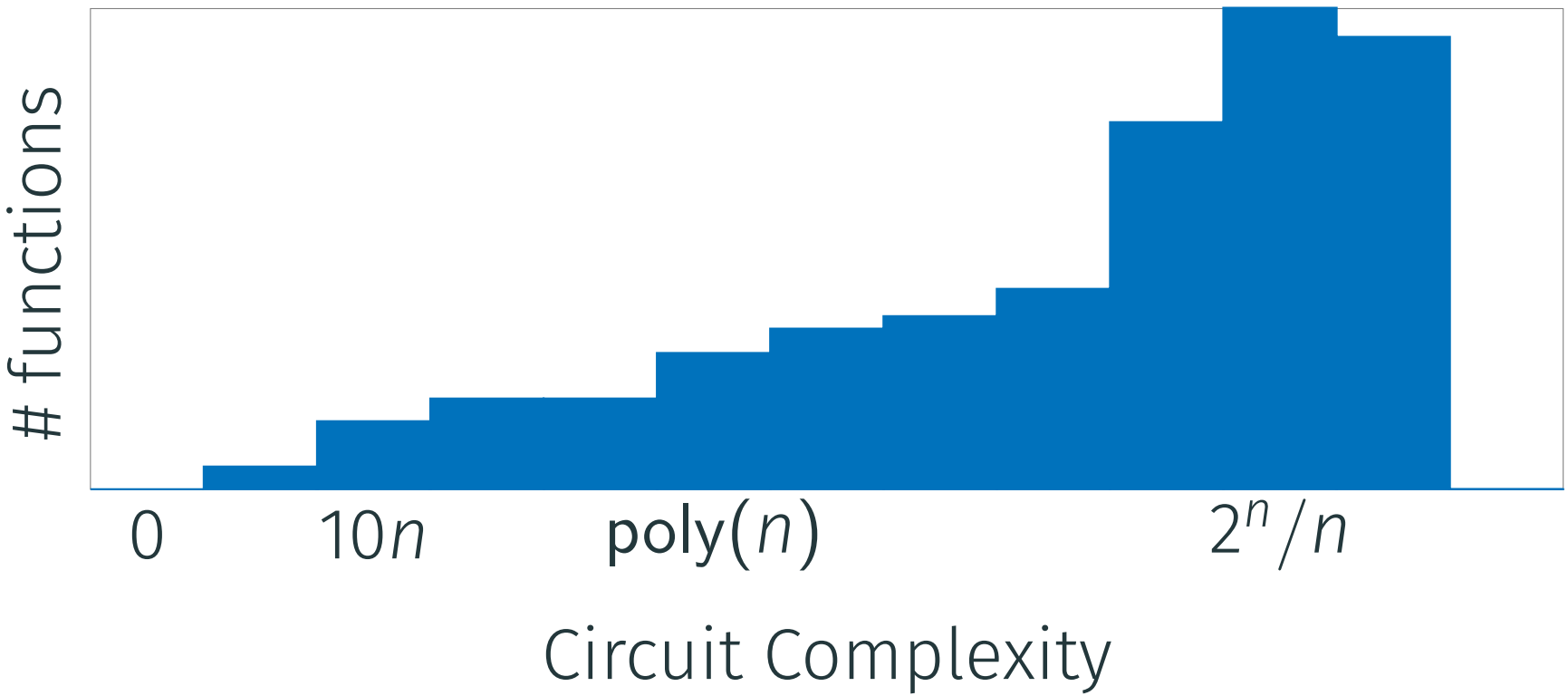

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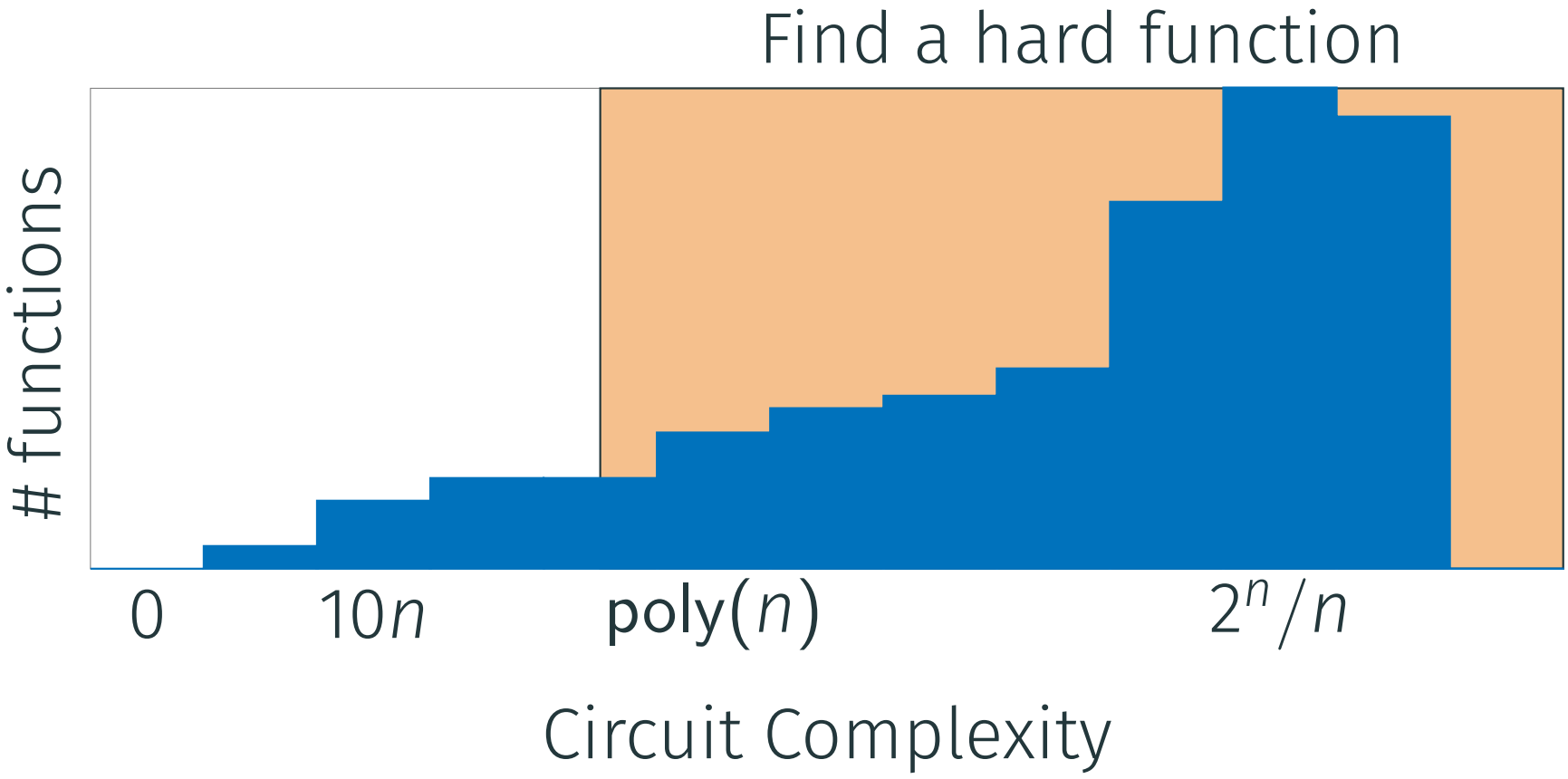
Find a hard function
functions
0
10n
poly(n)
2^n/n
Circuit Complexity

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Circuit Complexity

- Goal: Find a hard function
- Lower bounds: what functions are hard
- Upper bounds: what functions are easy

Circuit Upper Bound. Proof

Upper Bound [Lup1958]

Any function can be computed by a circuit of size

$$\leq 10 \cdot 2^n - 4$$

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$$f(x_1, \ldots, x_n) = \begin{cases} f(1, x_2, \ldots, x_n), & \text{if } x_1 = 1 \\ f(0, x_2, \ldots, x_n), & \text{if } x_1 = 0 \end{cases}$$

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$$\mathsf{Size}(n) \leq 4 + 2\,\mathsf{Size}(n-1) = O(2^n)$$

Circuit Lower Bound. Proof

Lower Bound [Sha1949]

Almost all functions of n variables have circuit size

$$\geq 2^n/(10n)$$