Gems of TCS

Linear Programming

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September 26, 2022

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- Optimization problems: among all solutions satisfying certain constraints find optimal one
- Find shortest cycle through all vertices
- Find optimal coloring
- Find minimum vertex cover
- Linear programming: class of optimization problems where constraints and optimization criterion are linear functions

Avoiding Scurvy

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- Orange has 100gm of vitamin C,
  grapefruit has 150gm of vitamin C,
  maximize daily vitamin C intake.

Avoiding Scurvy. Plot

$$\max 2x + 3y$$

$$x + y \leq 2$$

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$$x \geq 0$$

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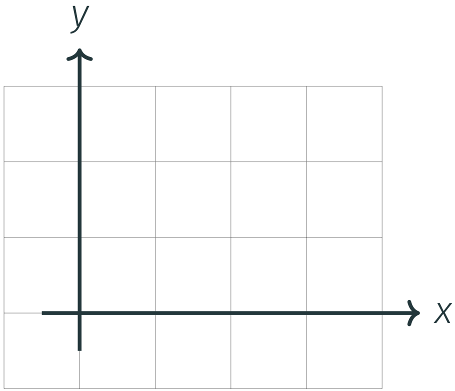

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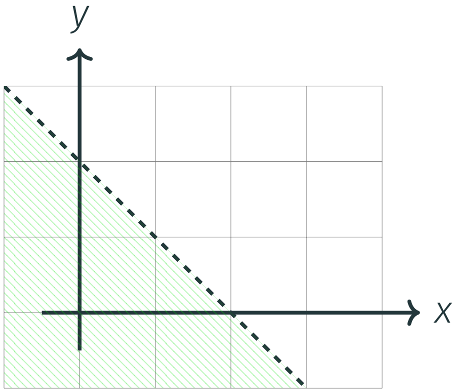

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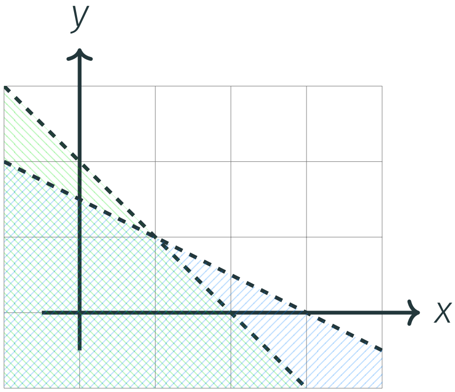

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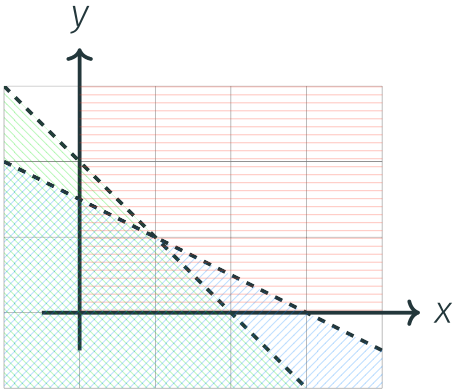

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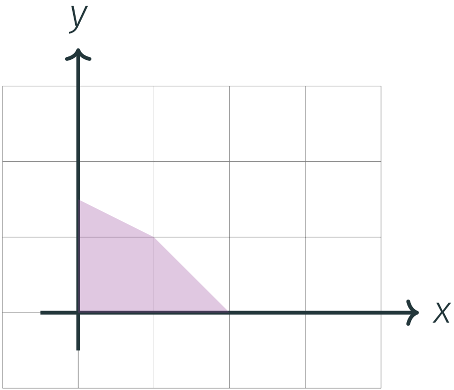

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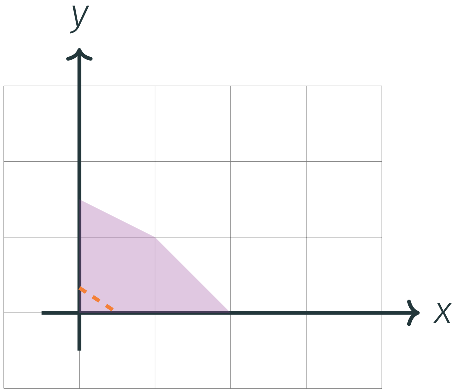

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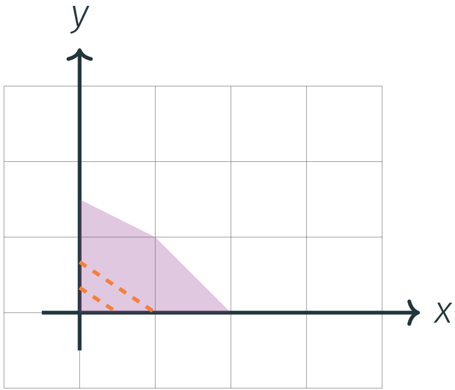

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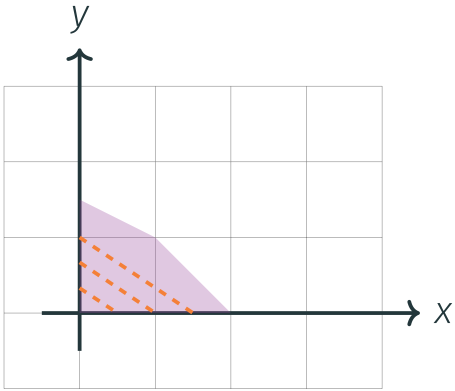

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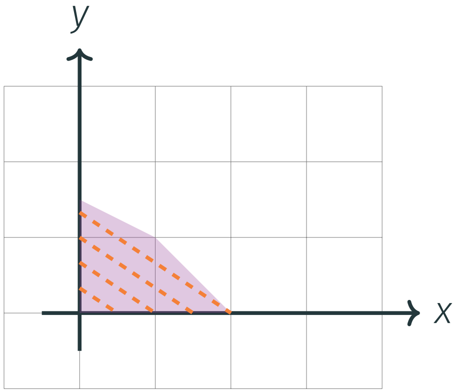

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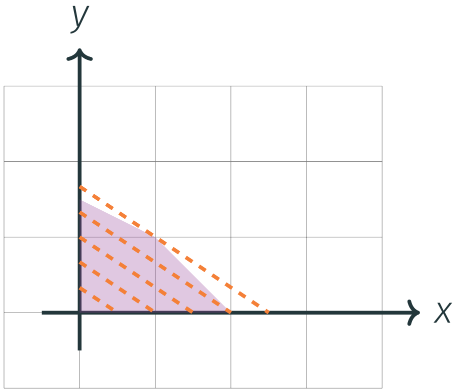

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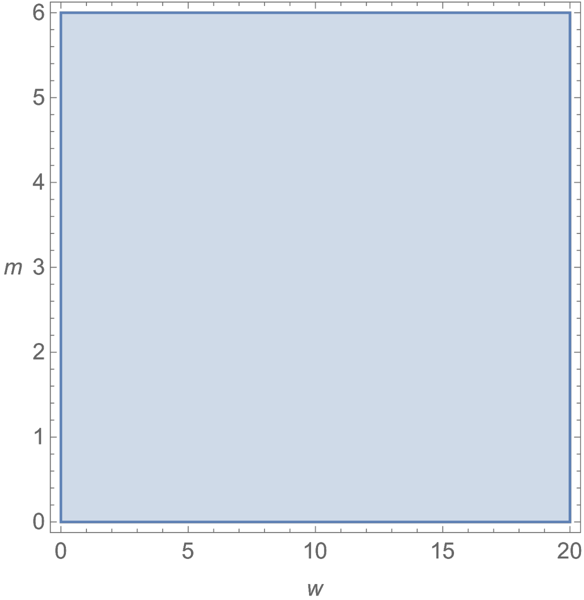
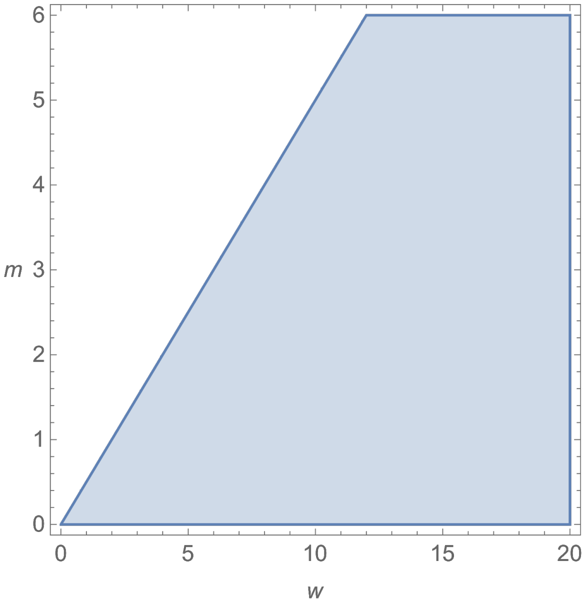
- We have 6 machines and 20 workers
- A machine takes two workers to operate
- Each machine produces 20 chocolates/hour, each worker produces 5 chocolates/hour
- We need to produce at most 100 chocolates/hour
- Each chocolate costs \$10, each worker gets \$40 per hour

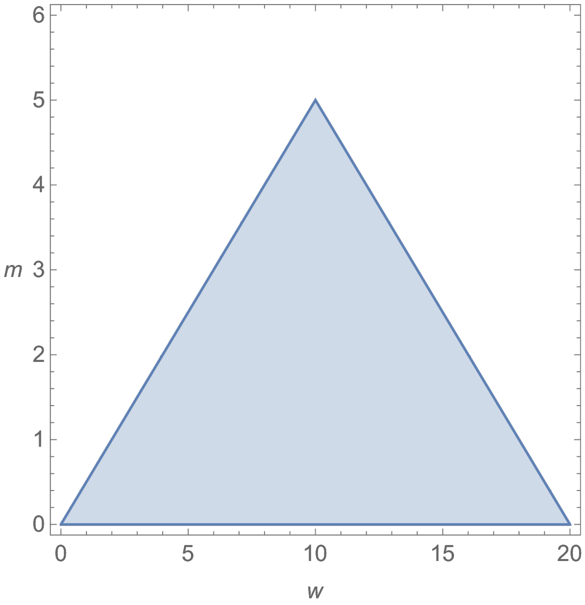
Workers and Machines



Two Workers Operate a Machine



Chocolate Demand



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- Find a linear function $h(a_1, \dots, a_d)$ s.t.
    - $h(a_1, \dots, a_d) < 0$ for all spam emails
    - $h(a_1, \dots, a_d) > 0$ for all ham emails

Linear Programming

Linear Programming

- Find real numbers $x_1, \ldots, x_n$ that satisfy linear constraints

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\geq b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &\geq b_2 \\
&\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\geq b_m
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- So that linear objective is maximized

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Equivalent Formulations

- Turn minimization problem into maximization problem:

$$\begin{aligned} \text{min} \quad & c_1x_1 + c_2x_2 + \ldots - c_nx_n \\ \text{max} \quad & -c_1x_1 - c_2x_2 - \ldots - c_nx_n \end{aligned}$$

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- Turn $\leq$ into $\geq$:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$
$$-a_{11}x_1 - a_{12}x_2 - \ldots - a_{1n}x_n \geq -b_1$$

Equivalent Formulations

- Turn $=$ into $\geq$:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

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Input is a matrix $A \in \mathbb{R}^{m \times n}$, and vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$

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$$\text{maximize } cx = \begin{bmatrix} c_1 & \dots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = c_1x_1 + \dots c_nx_n$$

History of Linear Programming

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- Kantorovich and Koopmans won Nobel Prize in Economics in 1971
- Dantzig's algorithm is "One of top 10 algorithms of the 20th century"

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Theorem

A linear function takes its maximum and minimum values on vertices

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- Start at any vertex

Simplex Method

Theorem

A linear function takes its maximum and minimum values on vertices

- Start at any vertex
- While there is an adjacent vertex with higher profit
    - Move to that vertex

Corner Cases

- No solutions

Corner Cases

- No solutions

- Unbounded profit

Algorithms for Linear Programming

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Ellipsoid Method