Gems of TCS

Gödel's Incompleteness

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October 12, 2022

Gödel's Incompleteness Theorem

Axiomatization of Math

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- Any proof could be (in principle) traced back to this set of axioms

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- Induction

Naive Set Theory

- Set
- Membership in a Set
- Empty Set
- Equality

Russell's paradox

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The barber is the ”one who shaves all those, and those only, who do not shave themselves”. The question is, does the barber shave himself?

Principia Mathematica

ZFC

Gödel's Incompleteness Theorem

Any attempt to axiomatize all of mathematics is guaranteed to fail

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- HALT is undecidable (Lecture 13)