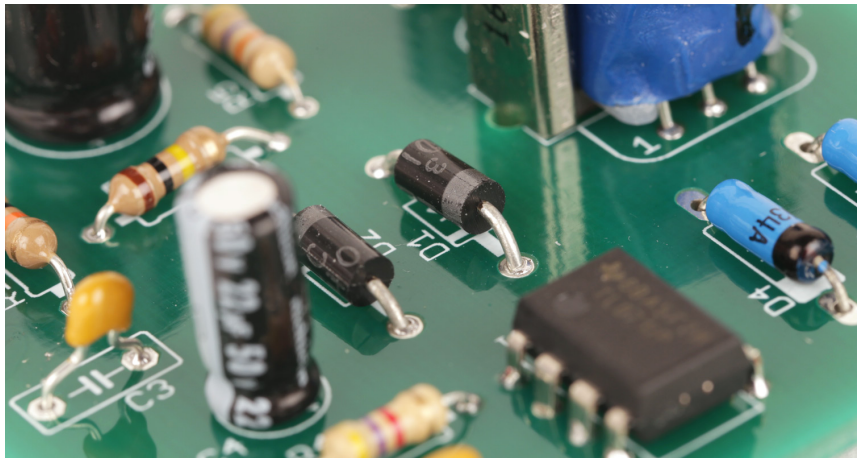


GEMS OF TCS

CIRCUIT COMPLEXITY II

Sasha Golovnev

October 24, 2022



The main open problem in Computer Science

Is **P** equal to **NP**?

The main open problem in Computer Science

Is **P** equal to **NP**?

- If **P=NP**, then all search problems can be solved in polynomial time.

The main open problem in Computer Science

Is **P** equal to **NP**?

- If **P=NP**, then all search problems can be solved in polynomial time.
- If **P $\neq$ NP**, then there exist search problems that cannot be solved in polynomial time.

BOOLEAN CIRCUITS

$$f: \{0, 1\}^n \rightarrow \{0, 1\}$$

$$g_1 = \neg x_1$$

$$g_2 = x_2 \wedge x_3$$

$$g_3 = g_1 \vee g_2$$

$$g_4 = g_2 \vee 1$$

$$g_5 = g_3 \wedge g_4$$

BOOLEAN CIRCUITS

$$f: \{0, 1\}^n \rightarrow \{0, 1\}$$

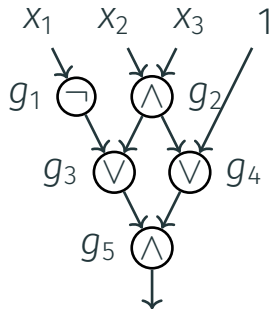
$$g_1 = \neg x_1$$

$$g_2 = x_2 \wedge x_3$$

$$g_3 = g_1 \vee g_2$$

$$g_4 = g_2 \vee 1$$

$$g_5 = g_3 \wedge g_4$$



EXPONENTIAL BOUNDS

Lower Bound [Sha1949]

Almost all functions of n variables have circuit size

$$\geq 2^n / n$$

EXPONENTIAL BOUNDS

Lower Bound [Sha1949]

Almost all functions of n variables have circuit size

$$\geq 2^n/n$$

Upper Bound [Lup1958]

Any function can be computed by a circuit of size

$$\leq 2^n/n$$

SYMMETRIC FUNCTIONS

$f: \{0, 1\}^n \rightarrow \{0, 1\}$ is **symmetric** if it depends only on the **number** of ones in the input, and **not** on **positions** of these ones.

SYMMETRIC FUNCTIONS

$f: \{0, 1\}^n \rightarrow \{0, 1\}$ is **symmetric** if it depends only on the **number** of ones in the input, and **not** on **positions** of these ones.

- $\text{And}(x) = 1$ iff $x_1 = \dots = x_n = 1$

SYMMETRIC FUNCTIONS

$f: \{0, 1\}^n \rightarrow \{0, 1\}$ is **symmetric** if it depends only on the **number** of ones in the input, and **not** on **positions** of these ones.

- $\text{And}(x) = 1$ iff $x_1 = \dots = x_n = 1$
- $\text{Or}(x) = 1$ iff $x_1 + \dots + x_n \geq 1$

SYMMETRIC FUNCTIONS

$f: \{0, 1\}^n \rightarrow \{0, 1\}$ is **symmetric** if it depends only on the **number** of ones in the input, and **not** on **positions** of these ones.

- $\text{And}(x) = 1$ iff $x_1 = \dots = x_n = 1$
- $\text{Or}(x) = 1$ iff $x_1 + \dots + x_n \geq 1$
- $\bigoplus(x) = 1$ iff $x_1 + \dots + x_n \equiv 1 \pmod{2}$

SYMMETRIC FUNCTIONS

$f: \{0, 1\}^n \rightarrow \{0, 1\}$ is **symmetric** if it depends only on the **number** of ones in the input, and **not** on **positions** of these ones.

- $\text{And}(x) = 1$ iff $x_1 = \dots = x_n = 1$
- $\text{Or}(x) = 1$ iff $x_1 + \dots + x_n \geq 1$
- $\bigoplus(x) = 1$ iff $x_1 + \dots + x_n \equiv 1 \pmod{2}$
- $\text{Mod}_k(x) = 1$ iff $x_1 + \dots + x_n \equiv 0 \pmod{k}$

SYMMETRIC FUNCTIONS

$f: \{0, 1\}^n \rightarrow \{0, 1\}$ is **symmetric** if it depends only on the **number** of ones in the input, and **not** on **positions** of these ones.

- $\text{And}(x) = 1$ iff $x_1 = \dots = x_n = 1$
- $\text{Or}(x) = 1$ iff $x_1 + \dots + x_n \geq 1$
- $\bigoplus(x) = 1$ iff $x_1 + \dots + x_n \equiv 1 \pmod{2}$
- $\text{Mod}_k(x) = 1$ iff $x_1 + \dots + x_n \equiv 0 \pmod{k}$
- $\text{Maj}(x) = 1$ iff $x_1 + \dots + x_n \geq n/2$

SYMMETRIC FUNCTIONS

$f: \{0, 1\}^n \rightarrow \{0, 1\}$ is **symmetric** if it depends only on the **number** of ones in the input, and **not** on **positions** of these ones.

- $\text{And}(x) = 1$ iff $x_1 = \dots = x_n = 1$
- $\text{Or}(x) = 1$ iff $x_1 + \dots + x_n \geq 1$
- $\bigoplus(x) = 1$ iff $x_1 + \dots + x_n \equiv 1 \pmod{2}$
- $\text{Mod}_k(x) = 1$ iff $x_1 + \dots + x_n \equiv 0 \pmod{k}$
- $\text{Maj}(x) = 1$ iff $x_1 + \dots + x_n \geq n/2$
- $\text{Th}_k(x) = 1$ iff $x_1 + \dots + x_n \geq k$

SYMMETRIC FUNCTIONS. EQUIV DEF

$f: \{0, 1\}^n \rightarrow \{0, 1\}$ is **symmetric** iff

$$f = g(x_1 + \dots + x_n)$$

for some $g: \{0, \dots, n\} \rightarrow \{0, 1\}$.

SYMMETRIC FUNCTIONS. EQUIV DEF

$f: \{0, 1\}^n \rightarrow \{0, 1\}$ is **symmetric** iff

$$f = g(x_1 + \dots + x_n)$$

for some $g: \{0, \dots, n\} \rightarrow \{0, 1\}$.

$f: \{0, 1\}^n \rightarrow \{0, 1\}$ is **symmetric** iff

$$f = h(\text{Sum}_n(x_1, \dots, x_n))$$

for some $h: \{0, 1\}^{\log n} \rightarrow \{0, 1\}$, where

$\text{Sum}_n: \{0, 1\}^n \rightarrow \{0, 1\}^{\log n}$.

COMPLEXITY OF Sum

$$\text{Sum}_3(x_1, x_2, x_3) \in \{0, 1\}^2$$

COMPLEXITY OF Sum

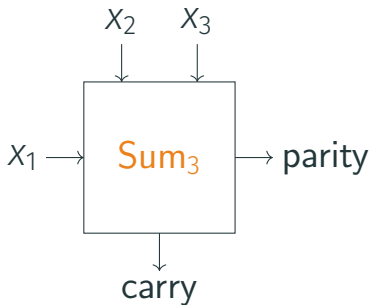
$$\text{Sum}_3(x_1, x_2, x_3) \in \{0, 1\}^2$$

$$\text{Sum}_3(x_1, x_2, x_3) = (\text{carry}, \text{parity})$$

COMPLEXITY OF Sum

$$\text{Sum}_3(x_1, x_2, x_3) \in \{0, 1\}^2$$

$$\text{Sum}_3(x_1, x_2, x_3) = (\text{carry}, \text{parity})$$

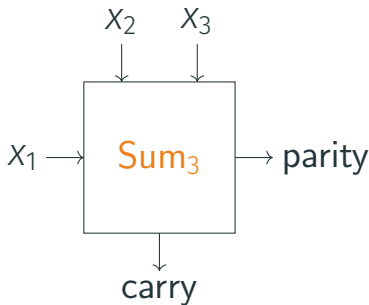


COMPLEXITY OF Sum

$$\text{Sum}_3(x_1, x_2, x_3) \in \{0, 1\}^2$$

$$\text{Sum}_3(x_1, x_2, x_3) = (\text{carry}, \text{parity})$$

$$\text{Size}(\text{Sum}_3) = O(1)$$

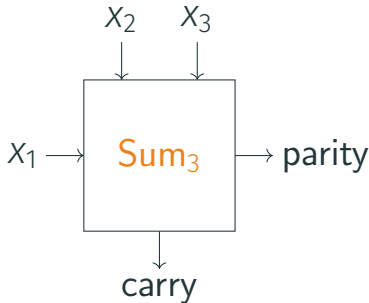


COMPLEXITY OF Sum

$$\text{Sum}_3(x_1, x_2, x_3) \in \{0, 1\}^2$$

$$\text{Sum}_3(x_1, x_2, x_3) = (\text{carry}, \text{parity})$$

$$\text{Size}(\text{Sum}_3) = O(1)$$



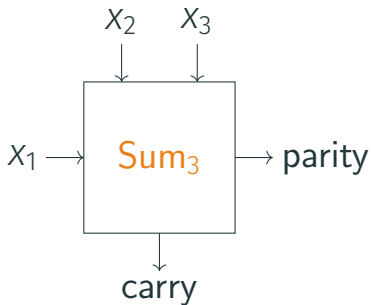
Sum₅?

COMPLEXITY OF Sum

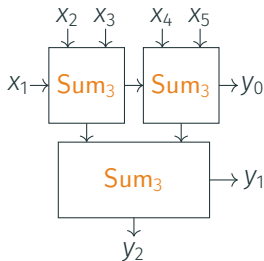
$$\text{Sum}_3(x_1, x_2, x_3) \in \{0, 1\}^2$$

$$\text{Sum}_3(x_1, x_2, x_3) = (\text{carry}, \text{parity})$$

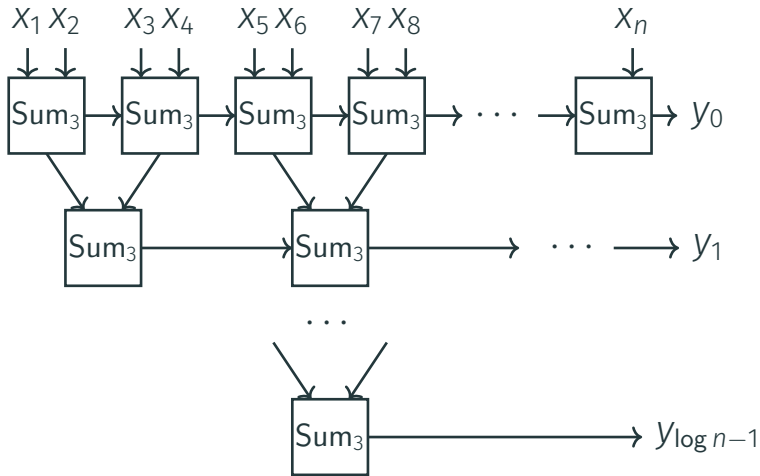
$$\text{Size}(\text{Sum}_3) = O(1)$$



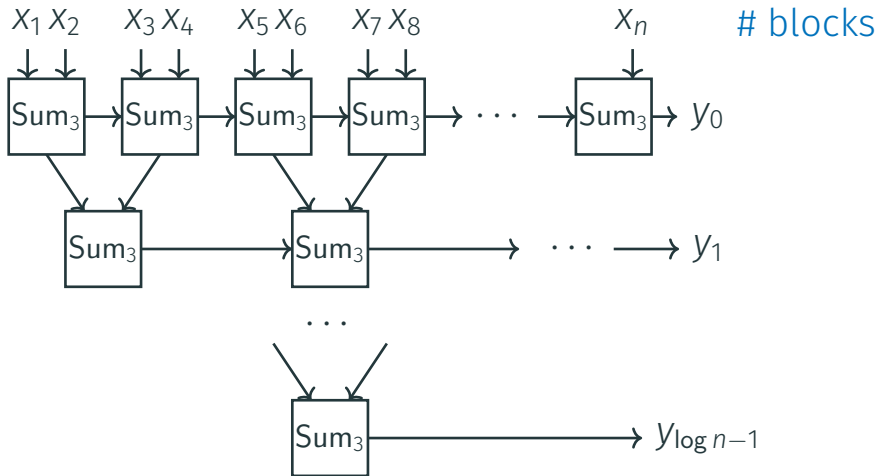
Sum₅?



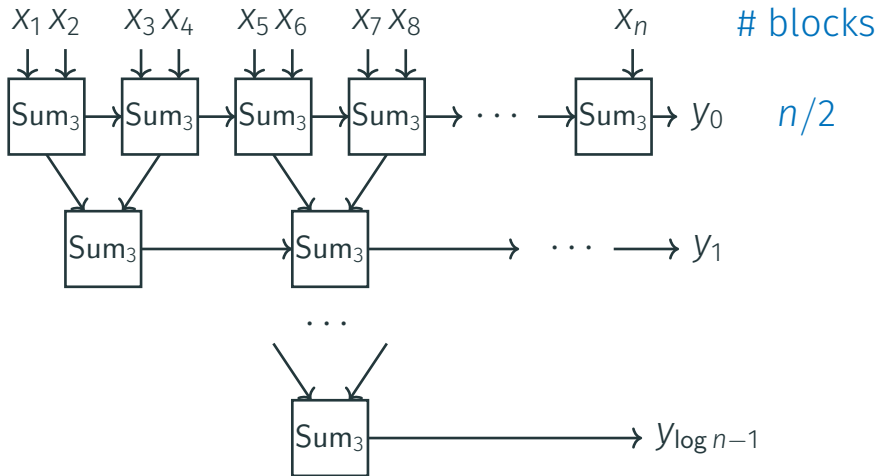
COMPLEXITY OF Sum_n



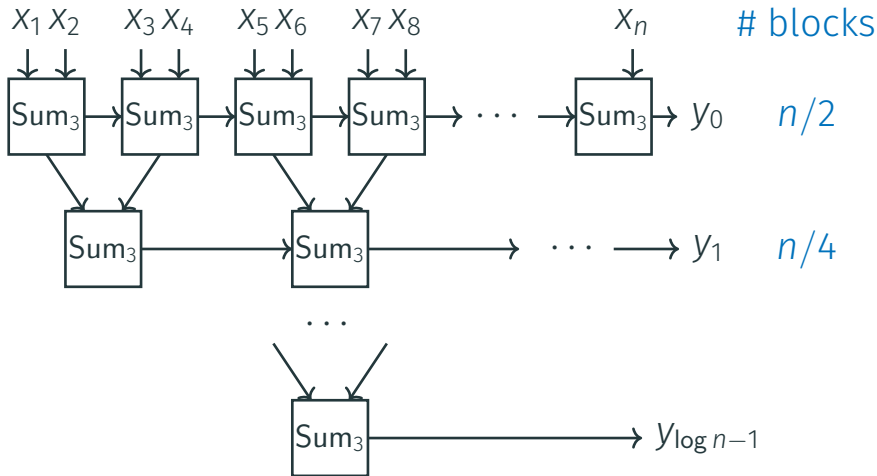
COMPLEXITY OF Sum_n



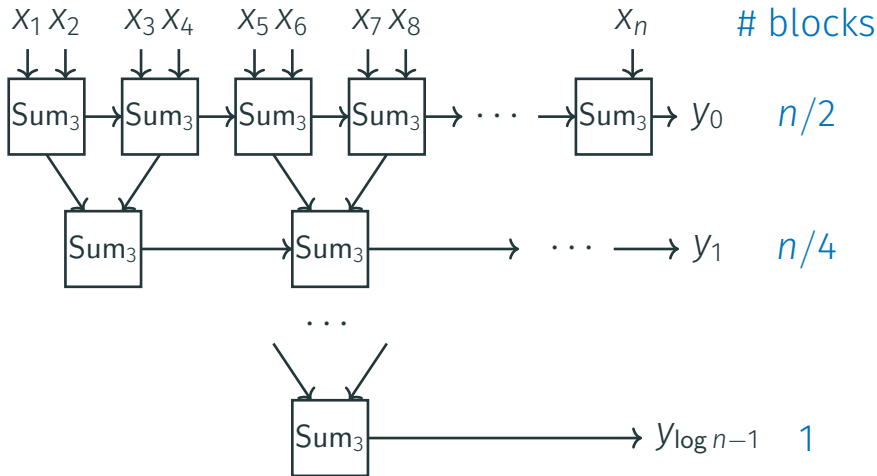
COMPLEXITY OF Sum_n



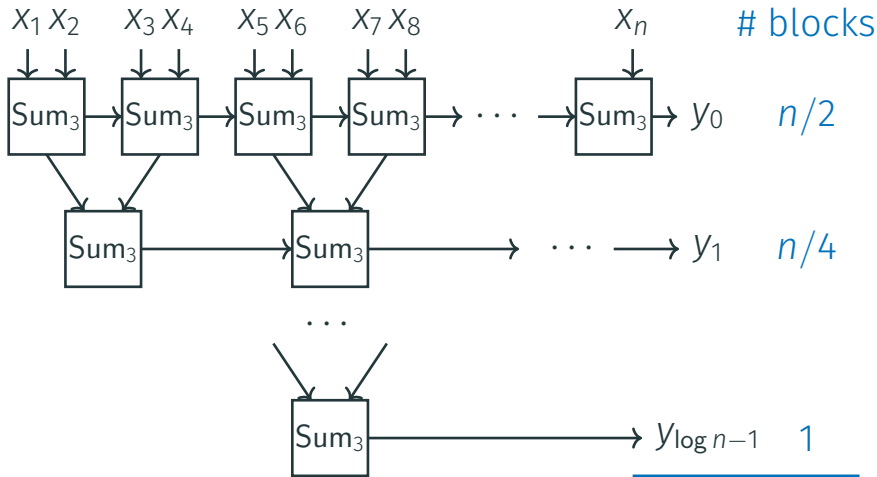
COMPLEXITY OF Sum_n



COMPLEXITY OF Sum_n

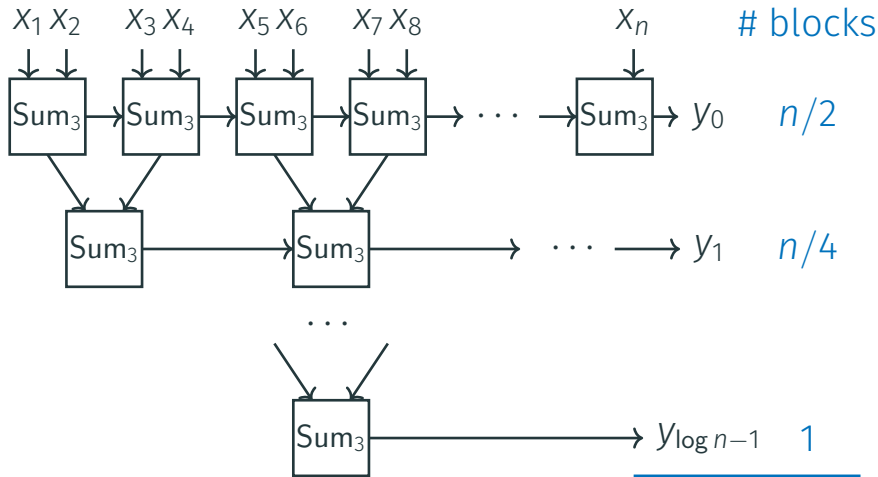


COMPLEXITY OF Sum_n



Total: $n - 1$

COMPLEXITY OF Sum_n



$$\text{Size}(\text{Sum}_n) < n \cdot \text{Size}(\text{Sum}_3) = O(n)$$

Total: $n - 1$

COMPLEXITY OF SYMMETRIC FUNCTIONS

Theorem

If $f: \{0, 1\}^n \rightarrow \{0, 1\}$ is a symmetric function, then

$$\text{Size}(f) \leq O(n).$$

COMPLEXITY OF SYMMETRIC FUNCTIONS

Theorem

If $f: \{0, 1\}^n \rightarrow \{0, 1\}$ is a **symmetric** function, then

$$\text{Size}(f) \leq O(n).$$

$$f = h(\text{Sum}_n(x_1, \dots, x_n)), \quad h: \{0, 1\}^{\log n} \rightarrow \{0, 1\}$$

COMPLEXITY OF SYMMETRIC FUNCTIONS

Theorem

If $f: \{0, 1\}^n \rightarrow \{0, 1\}$ is a **symmetric** function, then

$$\text{Size}(f) \leq O(n).$$

$$f = h(\text{Sum}_n(x_1, \dots, x_n)), \quad h: \{0, 1\}^{\log n} \rightarrow \{0, 1\}$$

$$\text{Size}(f) \leq \text{Size}(\text{Sum}_n) + \text{Size}(h)$$

COMPLEXITY OF SYMMETRIC FUNCTIONS

Theorem

If $f: \{0, 1\}^n \rightarrow \{0, 1\}$ is a **symmetric** function, then

$$\text{Size}(f) \leq O(n).$$

$$f = h(\text{Sum}_n(x_1, \dots, x_n)), \quad h: \{0, 1\}^{\log n} \rightarrow \{0, 1\}$$

$$\text{Size}(f) \leq \text{Size}(\text{Sum}_n) + \text{Size}(h)$$

$$\text{Size}(\text{Sum}_n) = O(n)$$

COMPLEXITY OF SYMMETRIC FUNCTIONS

Theorem

If $f: \{0, 1\}^n \rightarrow \{0, 1\}$ is a **symmetric** function, then

$$\text{Size}(f) \leq O(n).$$

$$f = h(\text{Sum}_n(x_1, \dots, x_n)), \quad h: \{0, 1\}^{\log n} \rightarrow \{0, 1\}$$

$$\text{Size}(f) \leq \text{Size}(\text{Sum}_n) + \text{Size}(h)$$

$$\text{Size}(\text{Sum}_n) = O(n)$$

$$\text{Size}(h) \leq 10 \cdot 2^{\log n} = O(n)$$

COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

- $k = 1, \text{Th}_1 = \text{Or}$

COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$

COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2$

COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$

COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$

COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”

COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

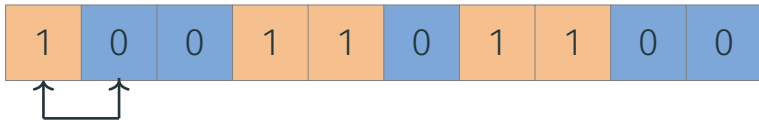
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

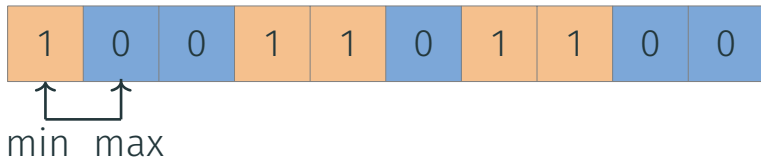
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

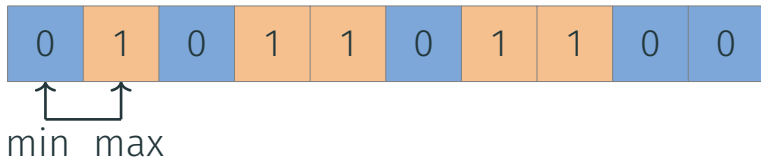
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

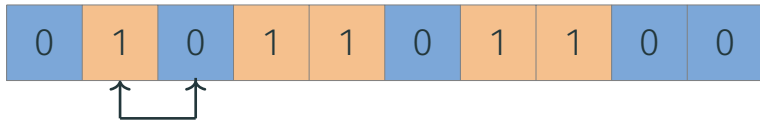
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

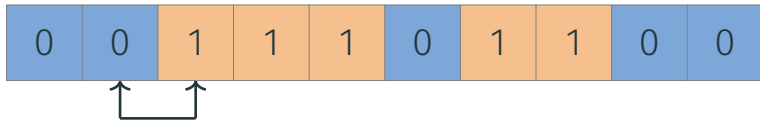
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

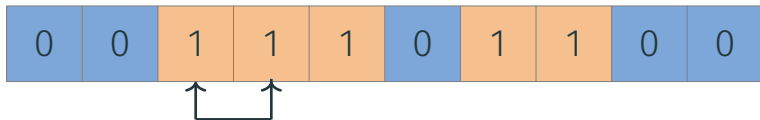
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

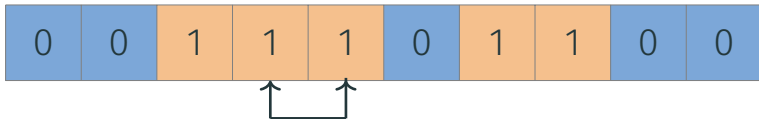
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

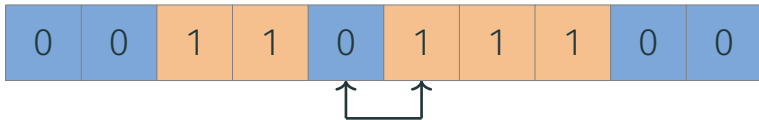
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

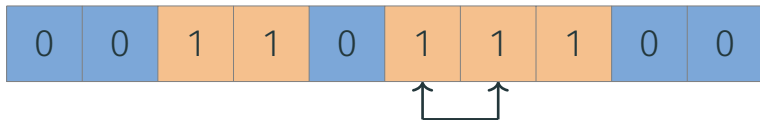
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

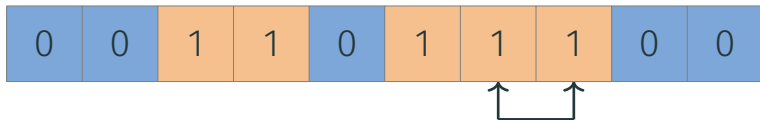
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

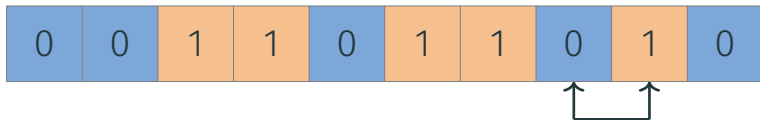
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

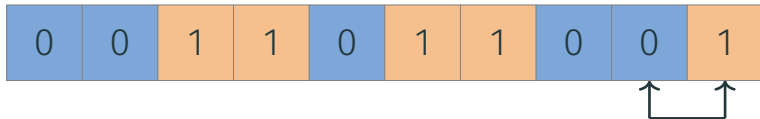
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

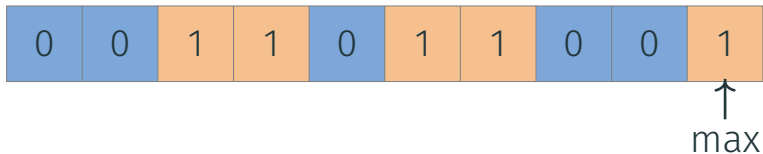
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

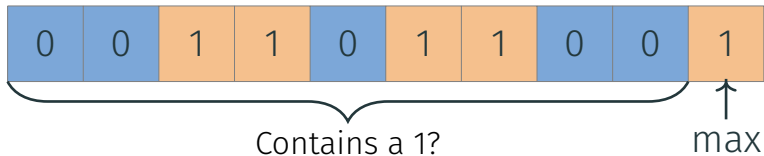
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

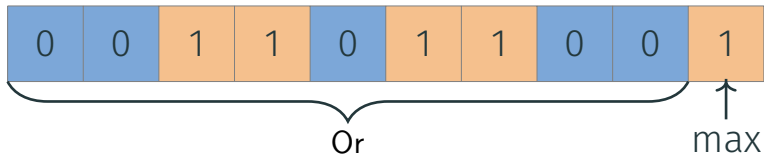
- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



COMPLEXITY OF THRESHOLD

$$\text{Th}_k(x) = 1 \text{ iff } x_1 + \dots + x_n \geq k.$$

- $k = 1, \text{Th}_1 = \text{Or}, \text{Size}(\text{Th}_1) = n - 1$
- $k = 2, \text{Size}(\text{Th}_2) = O(n)$
- $\text{Size}(\text{Th}_2) < 3n$
- Two rounds of “Bubble Sort”



Th₂. UPPER BOUND

$$\begin{array}{ccc} X_1 & \dots & X_{\sqrt{n}} \\ \vdots & & \vdots \\ X_{n-\sqrt{n}+1} & & X_n \end{array}$$

Th₂. UPPER BOUND

$$\begin{array}{ccc} x_1 & \dots & x_{\sqrt{n}} \\ \vdots & & \vdots \\ x_{n-\sqrt{n}+1} & & x_n \end{array}$$

$$\text{Th}_2(x_1, \dots, x_n) = 1 \text{ iff}$$

Th₂. UPPER BOUND

x_1	$\dots$	$x_{\sqrt{n}}$
$\vdots$		$\vdots$
$x_{n-\sqrt{n}+1}$		x_n

there are two cols with 1s

$$\text{Th}_2(x_1, \dots, x_n) = 1 \text{ iff } \text{OR}$$

there are two rows with 1s

Th₂. UPPER BOUND

$$\begin{array}{l} y_1 = \text{Or} \\ y_2 = \text{Or} \\ \vdots \\ y_{\sqrt{n}} = \text{Or} \end{array} \begin{array}{|c|c|c|} \hline x_1 & \cdots & x_{\sqrt{n}} \\ \hline & & \\ \hline \vdots & & \vdots \\ \hline & & \\ \hline x_{n-\sqrt{n}+1} & & x_n \\ \hline \end{array}$$

there are two cols with 1s

$$\text{Th}_2(x_1, \dots, x_n) = 1 \text{ iff } \text{OR}$$

there are two rows with 1s

Th₂. UPPER BOUND

$$\begin{array}{c}
 \begin{array}{c} z_1 \\ \parallel \\ \text{Or} \end{array}
 \begin{array}{c} z_2 \\ \parallel \\ \text{Or} \end{array}
 \begin{array}{c} \vdots \\ \parallel \\ \text{Or} \end{array}
 \begin{array}{c} z_{\sqrt{n}} \\ \parallel \\ \text{Or} \end{array} \\
 y_1 = \text{Or} \begin{array}{|c|} \hline x_1 \quad \dots \quad x_{\sqrt{n}} \\ \hline \end{array} \\
 y_2 = \text{Or} \begin{array}{|c|} \hline \vdots \quad \quad \quad \vdots \\ \hline \end{array} \\
 \vdots \\
 y_{\sqrt{n}} = \text{Or} \begin{array}{|c|} \hline x_{n-\sqrt{n}+1} \quad \quad x_n \\ \hline \end{array}
 \end{array}$$

there are two cols with 1s

$$\text{Th}_2(x_1, \dots, x_n) = 1 \text{ iff } \text{OR}$$

there are two rows with 1s

Th₂. UPPER BOUND

$$\begin{array}{c}
 z_1 \quad z_2 \quad \dots \quad z_{\sqrt{n}} \\
 \parallel \quad \parallel \quad \quad \parallel \\
 \text{Or} \quad \text{Or} \quad \quad \text{Or} \\
 y_1 = \text{Or} \quad \boxed{x_1 \quad \dots \quad x_{\sqrt{n}}} \\
 y_2 = \text{Or} \quad \boxed{\phantom{x_1 \quad \dots \quad x_{\sqrt{n}}}} \\
 \vdots \quad \quad \boxed{\phantom{x_1 \quad \dots \quad x_{\sqrt{n}}}} \\
 y_{\sqrt{n}} = \text{Or} \quad \boxed{x_{n-\sqrt{n}+1} \quad \dots \quad x_n}
 \end{array}$$

$$\text{Th}_2(x_1, \dots, x_n) = \text{Th}_2(y_1, \dots, y_{\sqrt{n}}) \text{ Or } \text{Th}_2(z_1, \dots, z_{\sqrt{n}})$$

Th₂. UPPER BOUND

$$\begin{array}{c}
 z_1 \quad z_2 \quad \dots \quad z_{\sqrt{n}} \\
 \parallel \quad \parallel \quad \quad \parallel \\
 \text{Or} \quad \text{Or} \quad \quad \text{Or} \\
 y_1 = \text{Or} \begin{array}{|c|} \hline x_1 \quad \dots \quad x_{\sqrt{n}} \\ \hline \end{array} \\
 y_2 = \text{Or} \begin{array}{|c|} \hline \vdots \quad \quad \quad \vdots \\ \hline \end{array} \\
 \vdots \\
 y_{\sqrt{n}} = \text{Or} \begin{array}{|c|} \hline x_{n-\sqrt{n}+1} \quad x_n \\ \hline \end{array}
 \end{array}$$

$$\text{Th}_2(x_1, \dots, x_n) = \text{Th}_2(y_1, \dots, y_{\sqrt{n}}) \text{ Or } \text{Th}_2(z_1, \dots, z_{\sqrt{n}})$$

$$\text{Size}(\text{Th}_2(n)) \leq 2n + 2 \text{Size}(\text{Th}_2(\sqrt{n}))$$

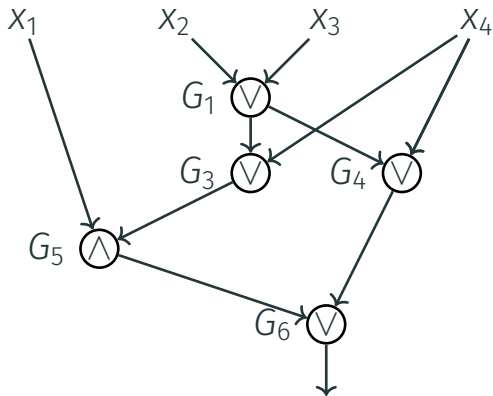
Th₂. UPPER BOUND

$$\begin{array}{cccc}
 & z_1 & z_2 & \dots & z_{\sqrt{n}} \\
 & \parallel & \parallel & & \parallel \\
 & \text{Or} & \text{Or} & & \text{Or} \\
 y_1 = \text{Or} & x_1 & & \dots & x_{\sqrt{n}} \\
 y_2 = \text{Or} & & & & \\
 \vdots & \vdots & & & \vdots \\
 y_{\sqrt{n}} = \text{Or} & x_{n-\sqrt{n}+1} & & & x_n
 \end{array}$$

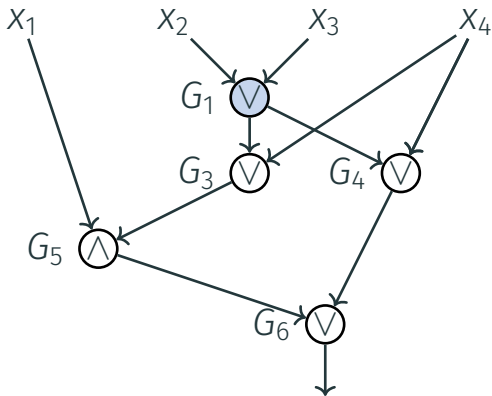
$$\text{Th}_2(x_1, \dots, x_n) = \text{Th}_2(y_1, \dots, y_{\sqrt{n}}) \text{ Or } \text{Th}_2(z_1, \dots, z_{\sqrt{n}})$$

$$\text{Size}(\text{Th}_2(n)) \leq 2n + 2 \text{Size}(\text{Th}_2(\sqrt{n})) \leq 2n + O(\sqrt{n})$$

Th₂. LOWER BOUND

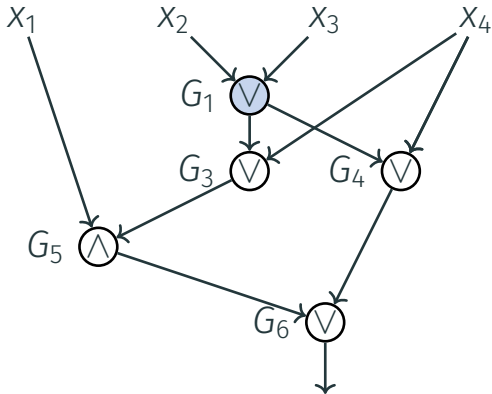


Th₂. LOWER BOUND



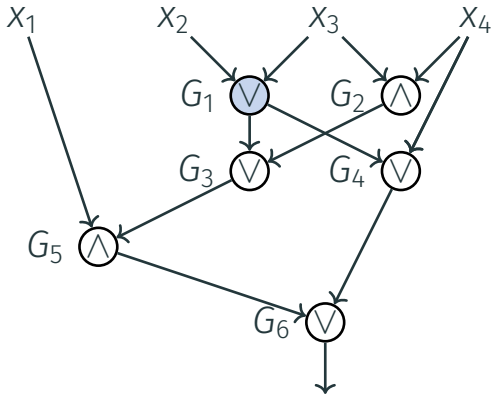
Th₂. LOWER BOUND

Case I:



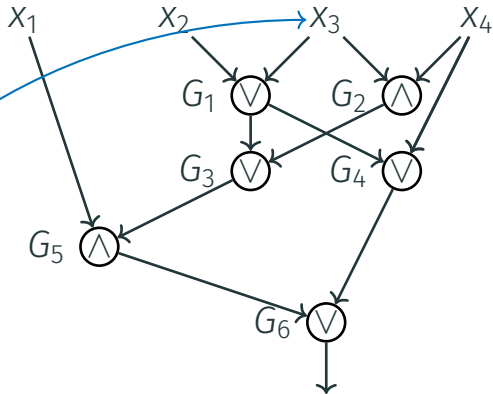
Th₂. LOWER BOUND

Case II:



Th₂. LOWER BOUND

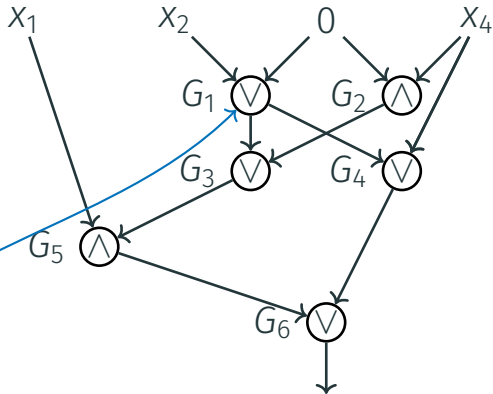
Case II:



assign $x_3 = 0$

Th₂. LOWER BOUND

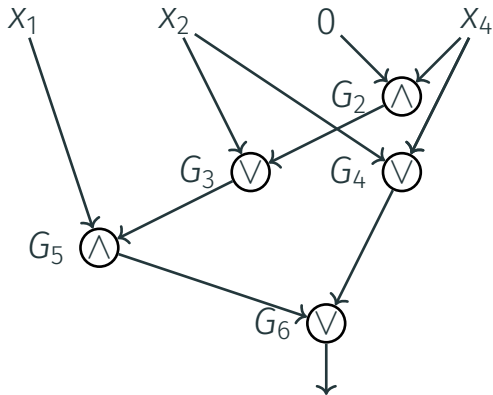
Case II:



G_1 now computes x_2

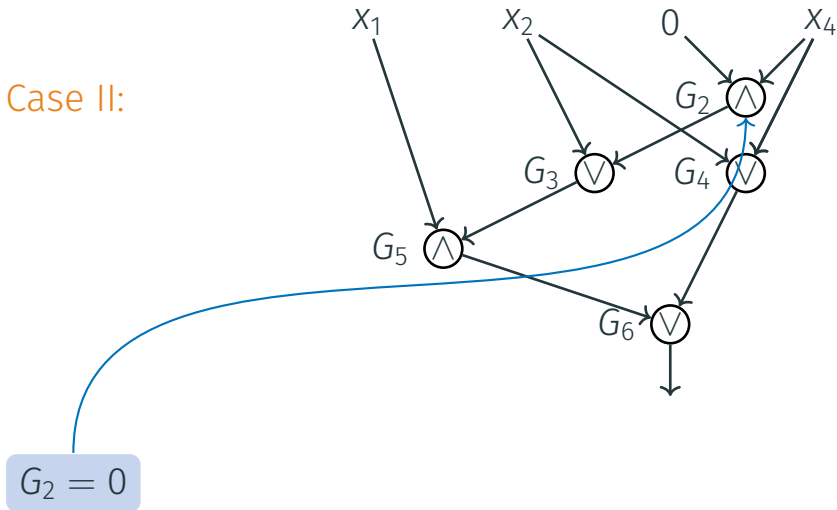
Th₂. LOWER BOUND

Case II:



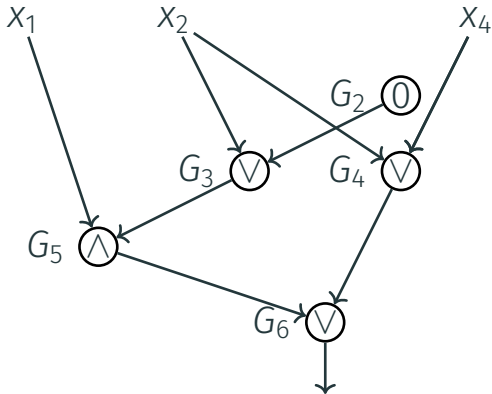
Th₂. LOWER BOUND

Case II:



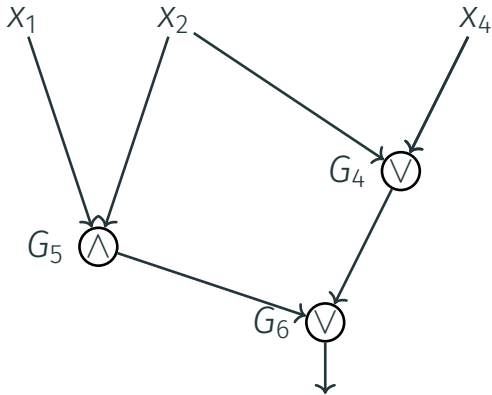
Th₂. LOWER BOUND

Case II:

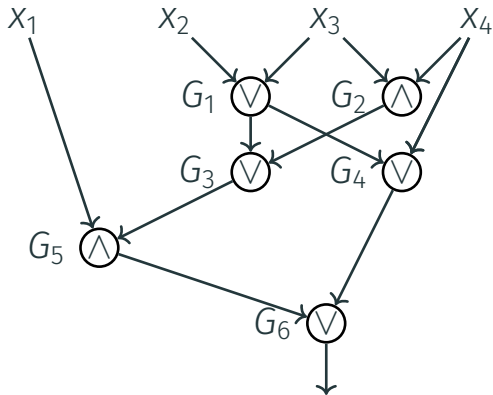


Th₂. LOWER BOUND

Case II:

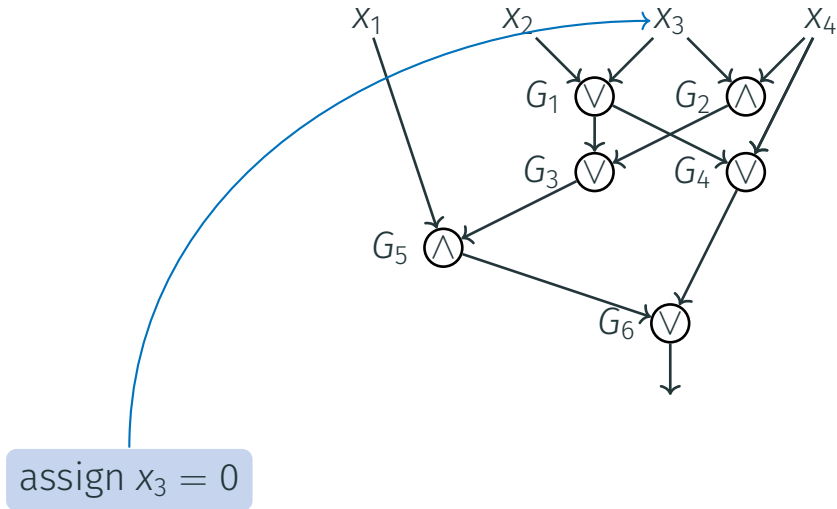


Th₂. LOWER BOUND

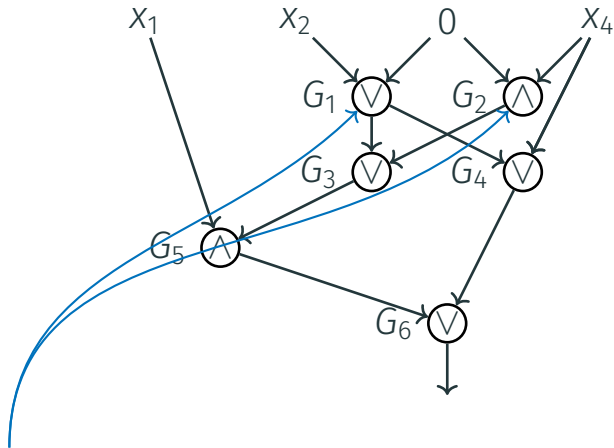


we start with circuit for Th_2^n

Th₂. LOWER BOUND

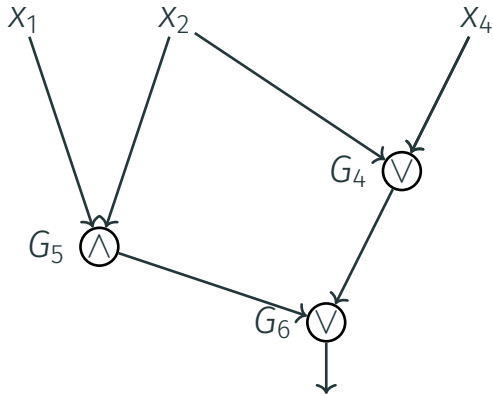


Th₂. LOWER BOUND



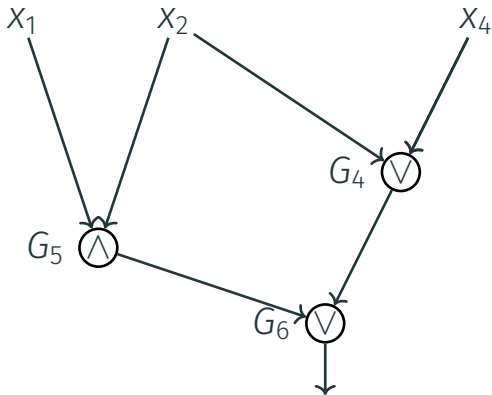
eliminate at least 2 gates

Th₂. LOWER BOUND



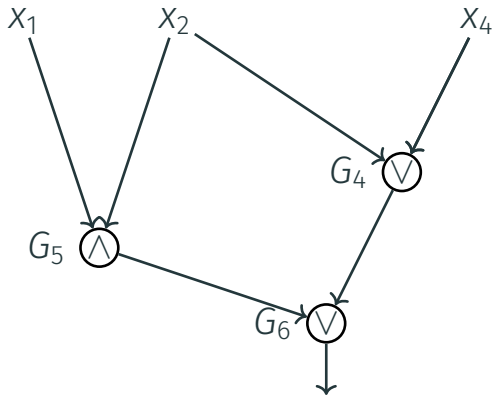
get a circuit for Th_2^{n-1}

Th₂. LOWER BOUND



$$\text{Size}(\text{Th}_2^n) \geq 2 + \text{Size}(\text{Th}_2^{n-1})$$

Th₂. LOWER BOUND



$$\text{Size}(\text{Th}_2^n) \geq 2 + \text{Size}(\text{Th}_2^{n-1}) \geq 2n - O(1)$$