Gems of TCS

Approximation Algorithms

Sasha Golovnev

August 29, 2022

Approximation Algorithms

- Optimal exact solution OPT (ex: shortest TSP cycle)

Approximation Algorithms

- Optimal exact solution OPT (ex: shortest TSP cycle)
- OPT is too hard to find (ex: NP-hard)

Approximation Algorithms

- Optimal exact solution OPT (ex: shortest TSP cycle)
- OPT is too hard to find (ex: NP-hard)
- A *k*-approximation algorithm finds a solution $\leq k \times \text{OPT}$

Approximation Algorithms

- Optimal exact solution OPT (ex: shortest TSP cycle)
- OPT is too hard to find (ex: **NP**-hard)
- A *k-approximation* algorithm finds a solution $\leq k \times \text{OPT}$
- Possibly efficiently! (ex: poly time)

Approximation Algorithms

- Optimal exact solution OPT (ex: shortest TSP cycle)
- OPT is too hard to find (ex: **NP**-hard)
- A *k*-approximation algorithm finds a solution $\leq k \times \mathsf{OPT}$
- Possibly efficiently! (ex: poly time)
- When do we use approximation algorithms?

Matchings

- A Matching in a graph is a set of edges without common vertices

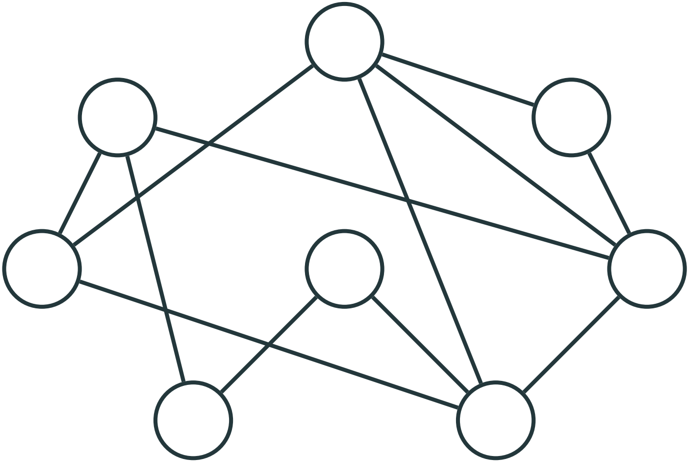
Matchings

- A Matching in a graph is a set of edges without common vertices
- A Maximal Matching is a matching which cannot be extended to a larger matching

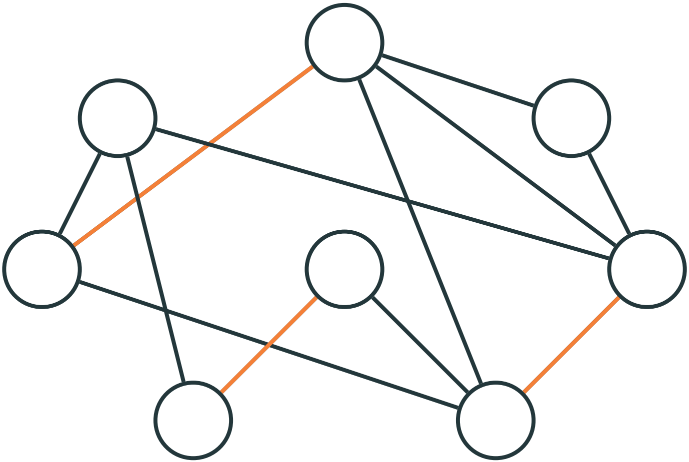
Matchings

- A Matching in a graph is a set of edges without common vertices
- A Maximal Matching is a matching which cannot be extended to a larger matching
- A Maximum Matching is a matching of the largest size

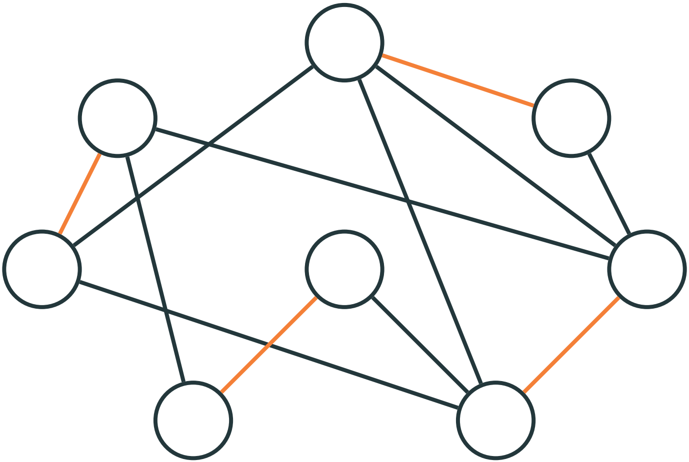
Matchings. Examples



Matchings. Examples



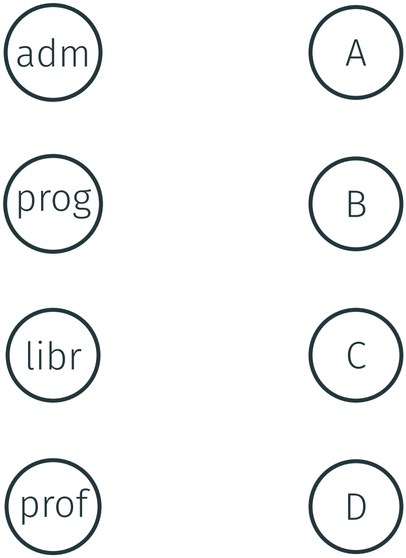
Matchings. Examples



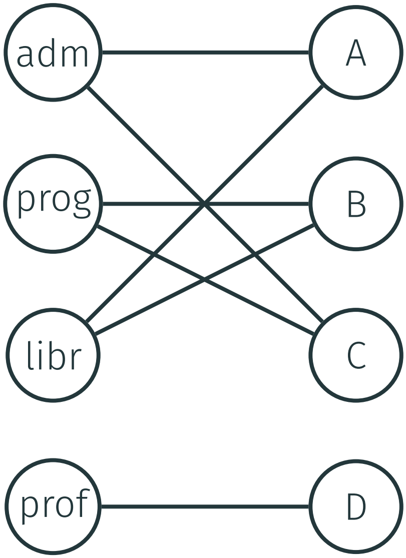
Job Assignment

	Alice	Ben	Chris	Diana
Administrator	+		+	
Programmer		+	+	
Librarian	+	+		
Professor				+

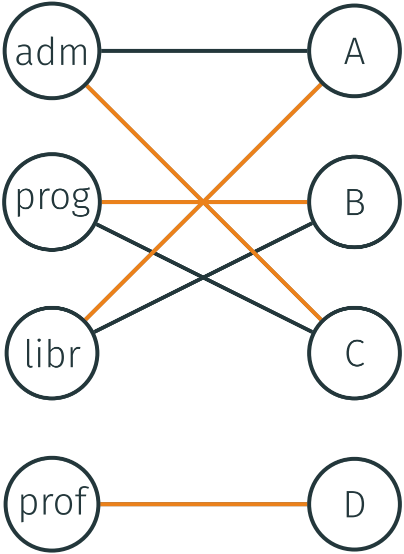
Job Assignment



Job Assignment



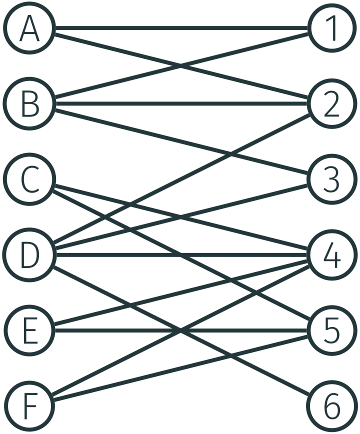
Job Assignment



Room Assignment

	R# 1	R# 2	R# 3	R# 4	R# 5	R# 6
Aaron	+	+				
Bianca	+	+	+			
Carol				+	+	
Dana		+	+	+		+
Emma				+	+	
Francis				+	+	

Room Assignment



Algorithms

Maximal Matching

Can be found in polynomial time by a greedy algorithm

Algorithms

Maximal Matching

Can be found in polynomial time by a greedy algorithm

Maximum Matching

Can be found in polynomial time by the blossom algorithm

Algorithms

Maximal Matching

Can be found in polynomial time by a greedy algorithm

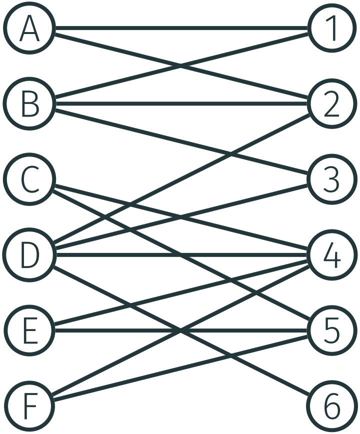
Maximum Matching

Can be found in polynomial time by the blossom algorithm

Minimum Weight Perfect Matching

Can be found in polynomial time by Edmonds' algorithm

Room Assignment



Vertex Cover

Vertex Covers

- A Vertex Cover of a graph G is a set of vertices C such that every edge of G is connected to some vertex in C.

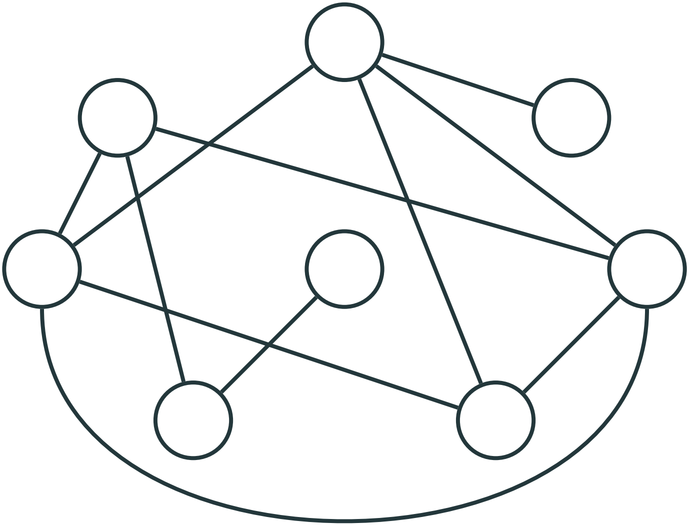
Vertex Covers

- A Vertex Cover of a graph G is a set of vertices C such that every edge of G is connected to some vertex in C.
- A Minimal Vertex Cover is a vertex cover which does not contain other vertex covers.

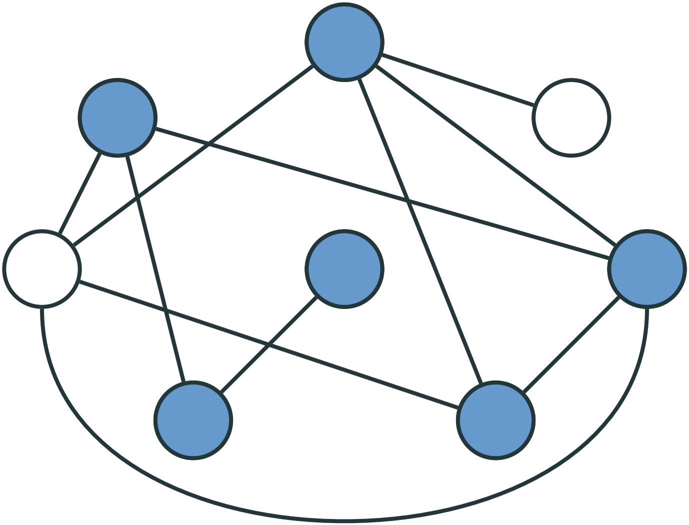
Vertex Covers

- A Vertex Cover of a graph G is a set of vertices C such that every edge of G is connected to some vertex in C.
- A Minimal Vertex Cover is a vertex cover which does not contain other vertex covers.
- A Minimum Vertex Cover is a vertex cover of the smallest size.

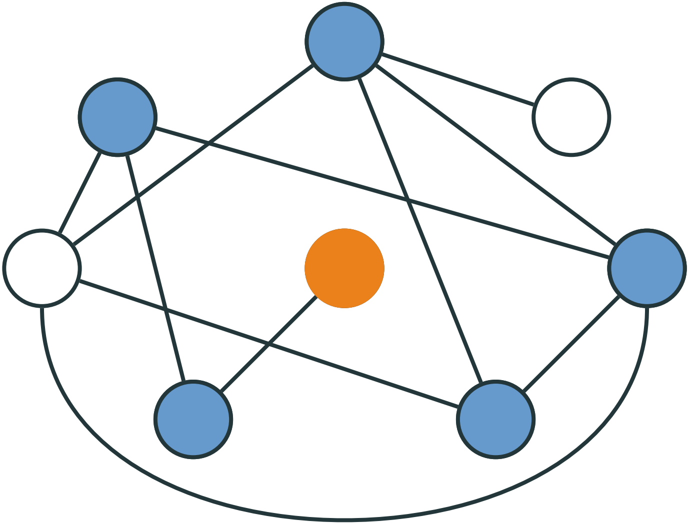
Vertex Covers: Examples



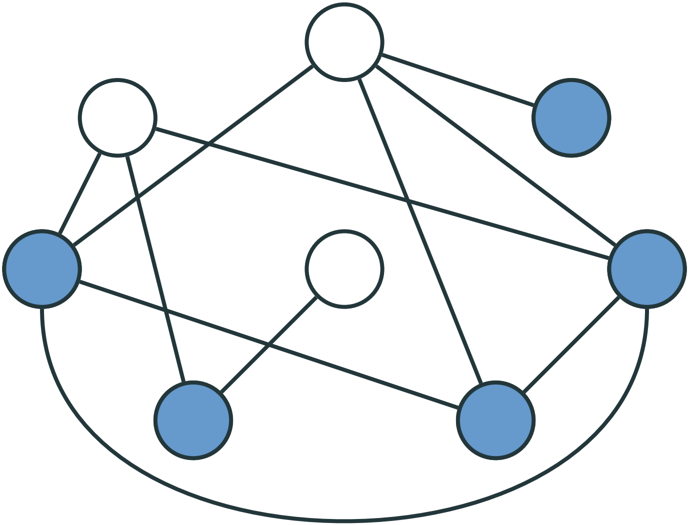
Vertex Covers: Examples



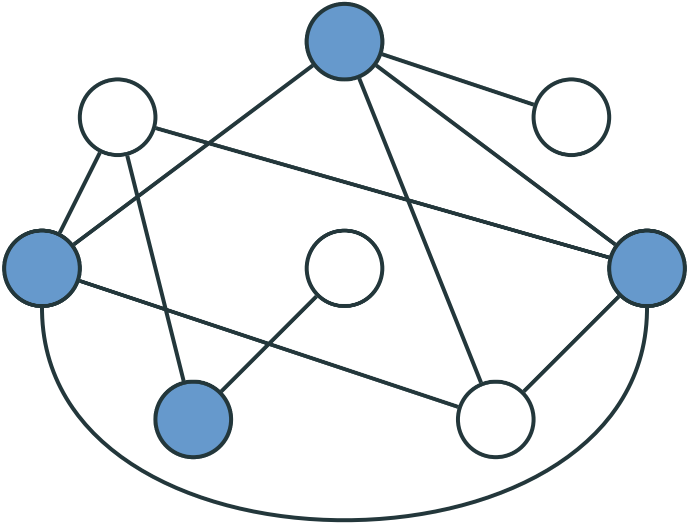
Vertex Covers: Examples



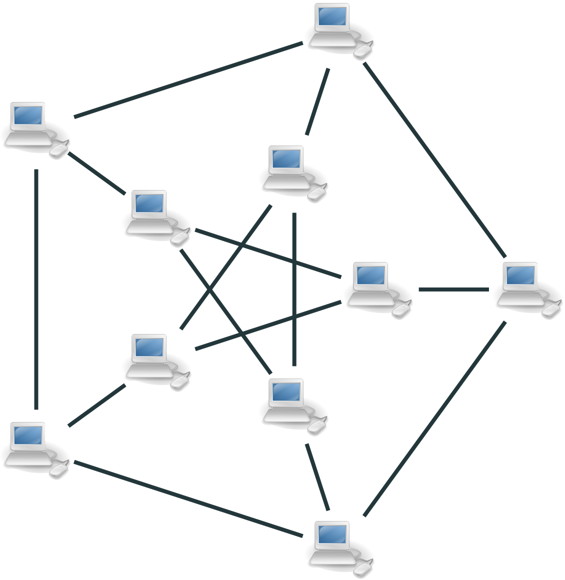
Vertex Covers: Examples



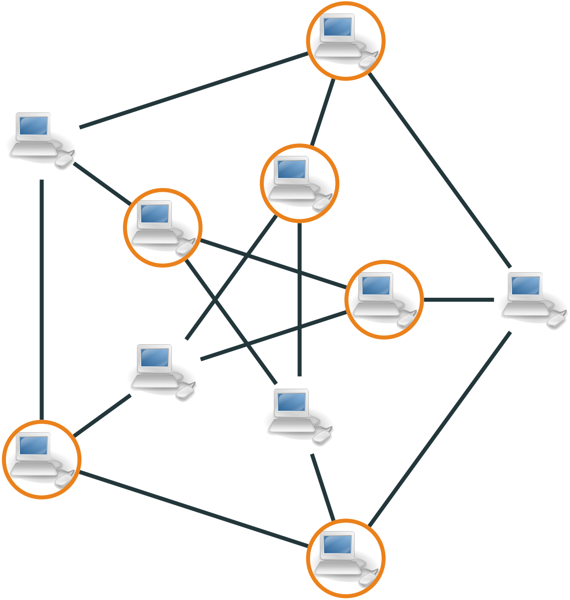
Vertex Covers: Examples



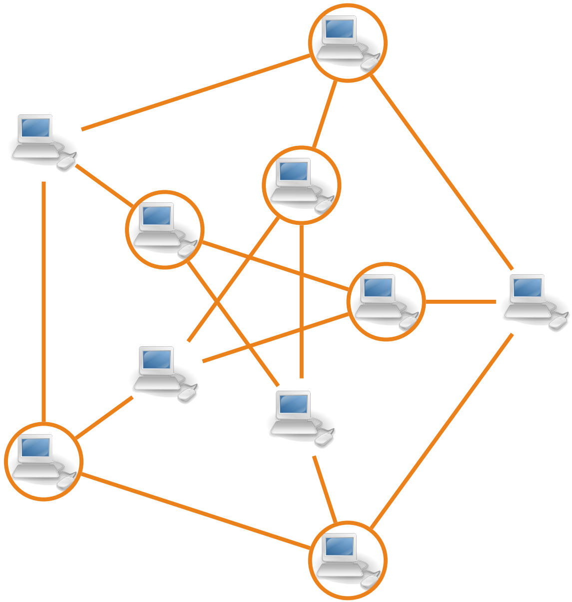
Antivirus System



Antivirus System



Antivirus System



Algorithms

Minimal Vertex Cover

Can be found in polynomial time by a greedy algorithm

Algorithms

Minimal Vertex Cover

Can be found in polynomial time by a greedy algorithm

Minimum Vertex Cover

Is **NP**-hard. We only know exponential-time algorithms

Approximation Algorithm

- $M \leftarrow$ maximal matching in G

Approximation Algorithm

- $M \leftarrow$ maximal matching in G
- return all vertices in M

Equivalent Algorithm

- $C \leftarrow \emptyset$

Equivalent Algorithm

- $C \leftarrow \emptyset$
- while $E \neq \emptyset$

Equivalent Algorithm

- $C \leftarrow \emptyset$
- while $E \neq \emptyset$
    - $\{u, v\} \leftarrow$ any edge from E

Equivalent Algorithm

- $C \leftarrow \emptyset$

- while $E \neq \emptyset$
    - $\{u, v\} \leftarrow$ any edge from E
    - add u, v to C

Equivalent Algorithm

- $C \leftarrow \emptyset$
- while $E \neq \emptyset$
    - $\{u, v\} \leftarrow$ any edge from E
    - add u, v to C
    - delete from E all edges incident to u or v
- return C

Proof

Lemma

This algorithm runs in polynomial time and is 2-approximate: it returns a vertex cover that is at most twice larger then a minimum vertex cover.

Final Remarks

- The analysis is tight: there are graphs with matchings twice larger than vertex covers

Final Remarks

- The analysis is tight: there are graphs with matchings twice larger than vertex covers

- No 1.99-approximation algorithm is known

Break

Matchings:

http://bit.ly/job-assignment

Vertex covers:

http://bit.ly/antivirus-system

Traveling Salesman

Approximation

- If $\mathsf{P} \neq \mathsf{NP}$, then there is no k-approximation algorithm for the general version of TSP for any constant k

Approximation

- If $\mathsf{P} \neq \mathsf{NP}$, then there is no k-approximation algorithm for the general version of TSP for any constant k
- Euclidean TSP: $w(u,v) = w(v,u)$ and $w(u,v) \leq w(u,z) + w(z,v)$

Approximation

- If $\mathsf{P} \neq \mathsf{NP}$, then there is no k-approximation algorithm for the general version of TSP for any constant k
- Euclidean TSP: $w(u, v) = w(v, u)$ and $w(u, v) \leq w(u, z) + w(z, v)$
- We will design a 2-approximation algorithm: it quickly finds a cycle that is at most twice longer than an optimal one

Definition

- A tree is a connected graph without cycles

Definition

- A tree is a connected graph without cycles
- A tree is a connected graph on n vertices with $n - 1$ edges

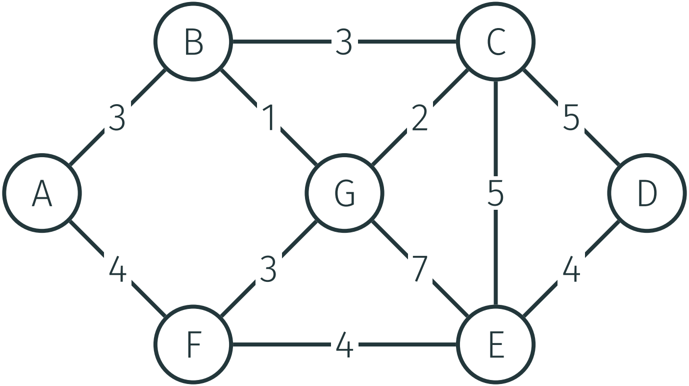
Definition

- A tree is a connected graph without cycles
- A tree is a connected graph on n vertices with $n - 1$ edges
- A Spanning Tree of a graph G is a subgraph of G that (i) is a tree and (ii) contains all vertices of G

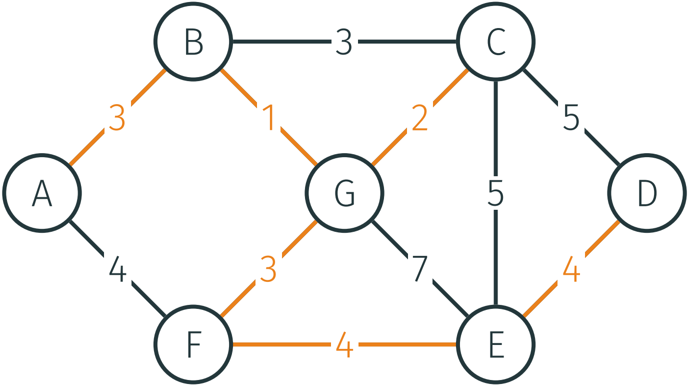
Definition

- A tree is a connected graph without cycles
- A tree is a connected graph on n vertices with $n - 1$ edges
- A Spanning Tree of a graph G is a subgraph of G that (i) is a tree and (ii) contains all vertices of G
- A Minimum Spanning Tree of a weighted graph G is a spanning tree of the smallest weight

Minimum Spanning Tree: Examples



Minimum Spanning Tree: Examples



Lemma

Let G be an undirected graph with non-negative edge weights. Then $\mathsf{MST}(G) \leq \mathsf{TSP}(G)$.

Minimum Spanning Trees

Lemma

Let G be an undirected graph with non-negative edge weights. Then $\mathsf{MST}(G) \leq \mathsf{TSP}(G)$.

Proof

By removing any edge from an optimum TSP cycle one gets a spanning tree of G.

Eulerian Cycle

An Eulerian cycle (or path) visits every edge exactly once

Eulerian Cycle

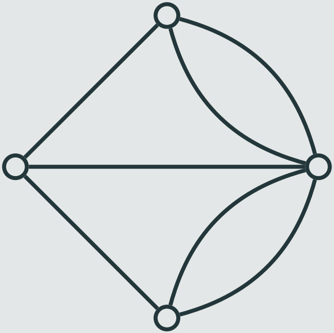
An Eulerian cycle (or path) visits every edge exactly once

Criteria

A connected undirected graph contains an Eulerian cycle, if and only if the degree of every node is even

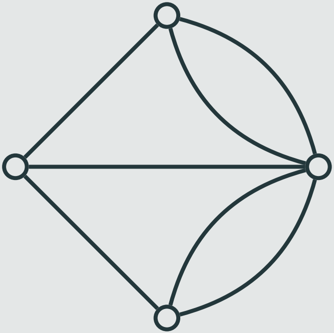
Example

Non-Eulerian graph

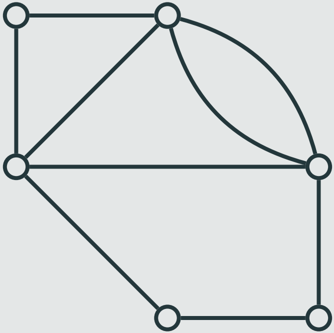


Example

Non-Eulerian graph



Eulerian graph



Algorithm

- $T \leftarrow$ minimum spanning tree of G

Algorithm

- $T \leftarrow$ minimum spanning tree of G
- $D \leftarrow T$ with each edge doubled

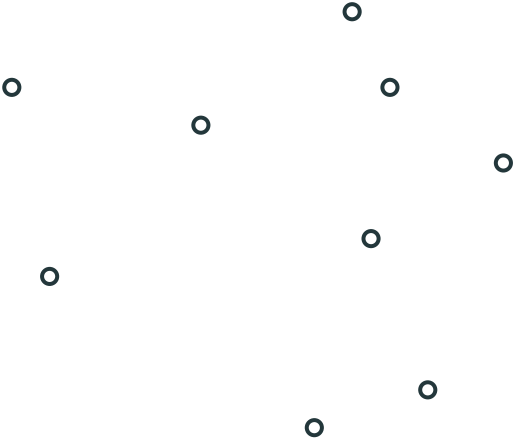
Algorithm

- $T \leftarrow$ minimum spanning tree of G
- $D \leftarrow T$ with each edge doubled
- find an Eulerian cycle C in D

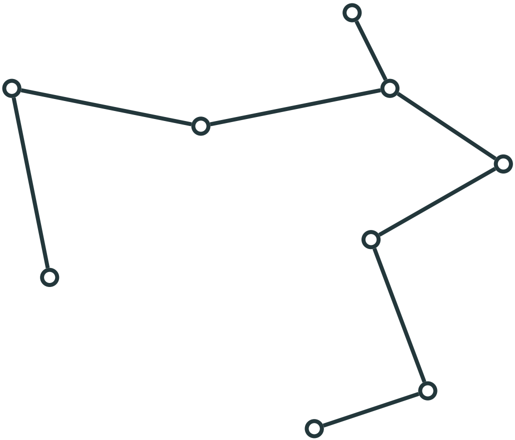
Algorithm

- $T \leftarrow$ minimum spanning tree of G
- $D \leftarrow T$ with each edge doubled
- find an Eulerian cycle C in D
- return a cycle that visits the nodes in the order of their first appearance in C

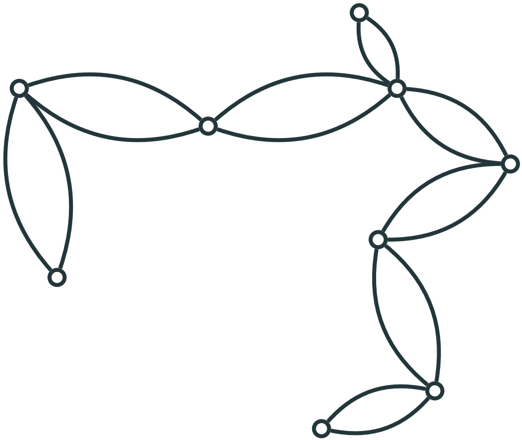
Example



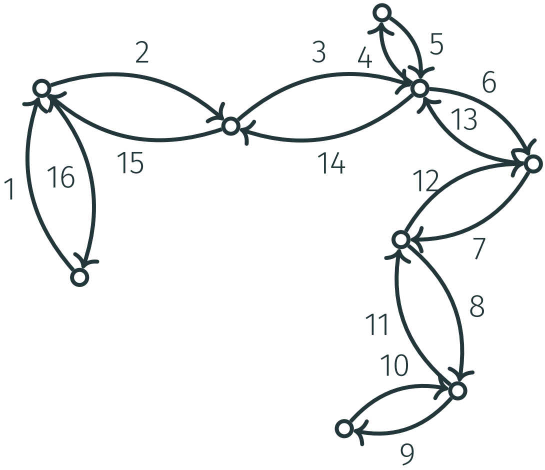
Example



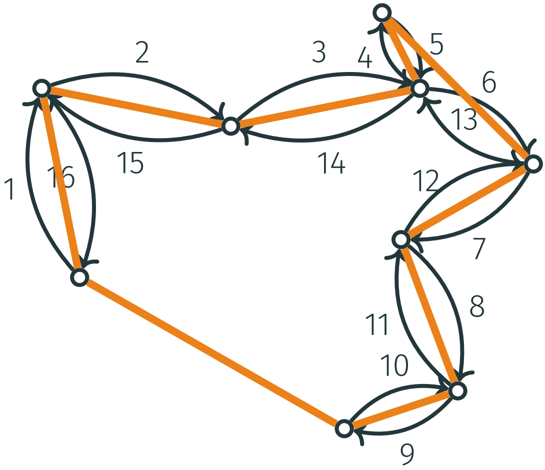
Example



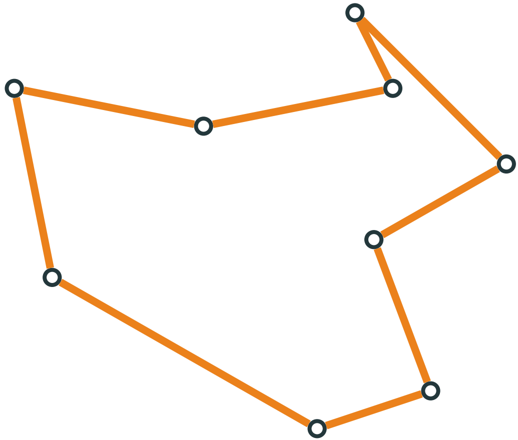
Example



Example



Example



Approximation Guarantee

Lemma

The algorithm is 2-approximate.

Approximation Guarantee

Lemma

The algorithm is 2-approximate.

Proof

- The total length of the MST $T \leq \text{OPT}$

Approximation Guarantee

Lemma

The algorithm is 2-approximate.

Proof

- The total length of the MST $T \leq \mathrm{OPT}$
- We start with Eulerian cycle of length $2|T|$

Approximation Guarantee

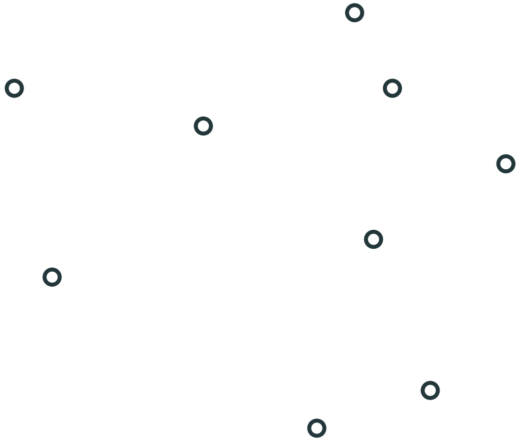
Lemma

The algorithm is 2-approximate.

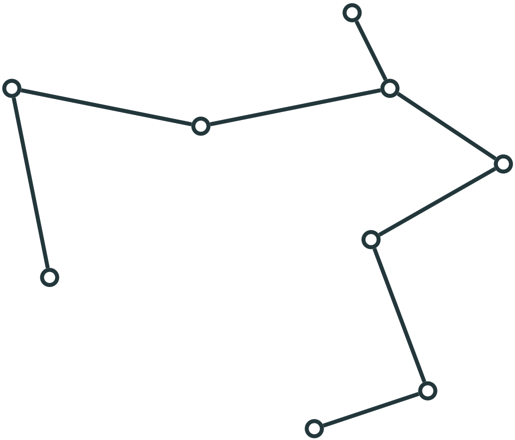
Proof

- The total length of the MST $T \leq \mathrm{OPT}$
- We start with Eulerian cycle of length $2|T|$
- Shortcuts can only decrease the total length

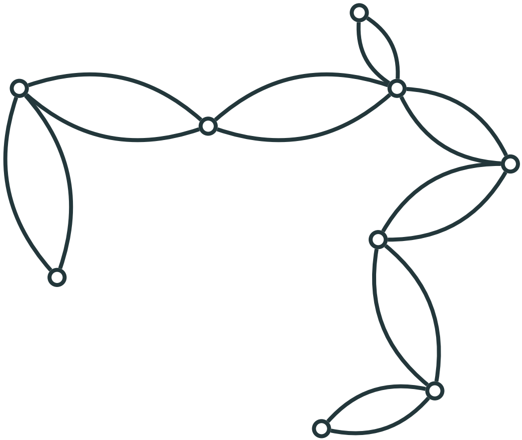
Improvement



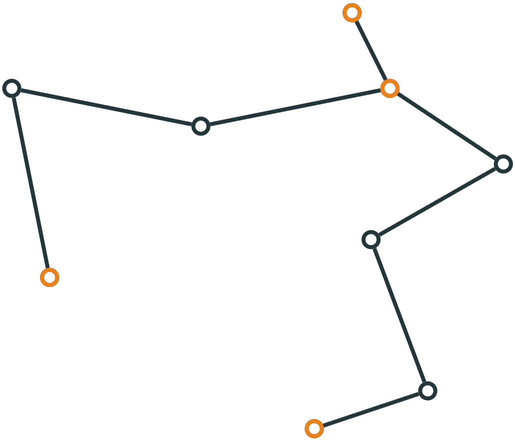
Improvement



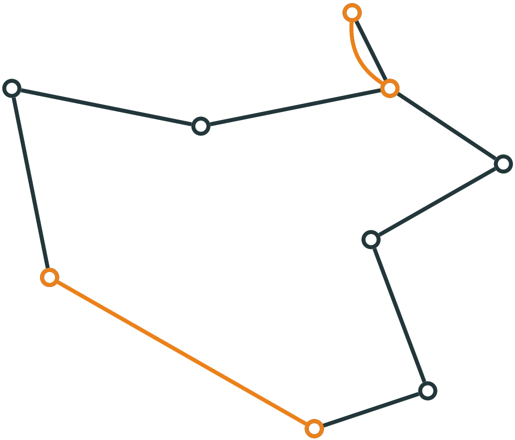
Improvement



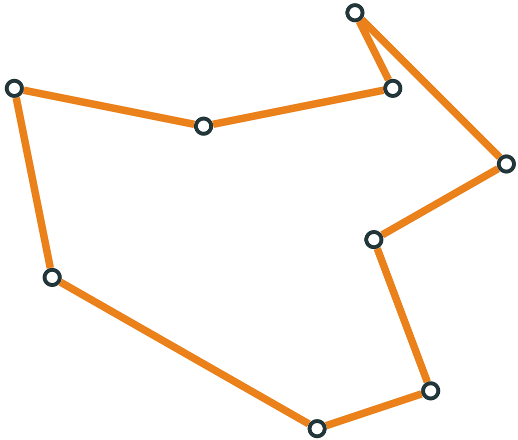
Improvement



Improvement



Improvement



Algorithm

- $T \leftarrow$ minimum spanning tree of G

Algorithm

- $T \leftarrow$ minimum spanning tree of G
- $M \leftarrow$ minimum weight perfect matching on odd-degree vertices of T

Algorithm

- $T \leftarrow$ minimum spanning tree of G
- $M \leftarrow$ minimum weight perfect matching on odd-degree vertices of T
- find an Eulerian cycle C in $T \cup M$

Algorithm

- $T \leftarrow$ minimum spanning tree of G
- $M \leftarrow$ minimum weight perfect matching on odd-degree vertices of T
- find an Eulerian cycle C in $T \cup M$
- return a cycle that visits the nodes in the order of their first appearance in C

Approximation Guarantee

Lemma

The algorithm is 3/2-approximate.

Approximation Guarantee

Lemma

The algorithm is 3/2-approximate.

Proof

- The total length of the MST $T \leq \mathrm{OPT}$

Approximation Guarantee

Lemma

The algorithm is 3/2-approximate.

Proof

- The total length of the MST $T \leq \mathrm{OPT}$
- The weight of the matching $M \leq \mathrm{OPT}/2$

Approximation Guarantee

Lemma

The algorithm is 3/2-approximate.

Proof

- The total length of the MST $T \leq \mathrm{OPT}$
- The weight of the matching $M \leq \mathrm{OPT}/2$
- Shortcuts can only decrease the total length

Final Remarks

- Euclidean TSP can be approximated to within any factor $(1 + \varepsilon)$

Final Remarks

- Euclidean TSP can be approximated to within any factor $(1 + \varepsilon)$

- The currently best known approximation algorithm for TSP with triangle inequality is has approximation factor of $3/2 - 10^{-36}$ (July 2020)