Problem 1 (Rigidity upper bound)**.** Let $\mathbb{F}$ be a field, and let a matrix $A \in \mathbb{F}^{n\times n}$ be written as

$$A = \begin{pmatrix} B & A_{12} \\ A_{21} & C \end{pmatrix} ,$$

where $B \in \mathbb{F}^{r\times r}, A_{12} \in \mathbb{F}^{r\times(n-r)}, A_{21} \in \mathbb{F}^{(n-r)\times r}, C \in \mathbb{F}^{(n-r)\times(n-r)}$.

Prove that if $\operatorname{rank}(B) = r$ and $C = A_{21}B^{-1}A_{12}$, then

$$\operatorname{rank}(A) = \operatorname{rank}(B) = r .$$

Problem 2 (Linear codes)**.** Prove that for every $\delta < 1/2$ and $\varepsilon > 0$, there exists a subspace $C \subseteq \mathbb{F}_2^n$ of dimension $k \geq n(1 - H(\delta) - \varepsilon)$ such that for all non-zero $x \in C$: $\|x\|_1 \geq \delta n$. Here $H(p)$ denotes the binary entropy function

$$H(p) = p \log_2 \frac{1}{p} + (1-p) \log_2 \frac{1}{1-p} .$$

In order to prove this, consider a greedy algorithm that sequentially adds k basis vectors which are δn-far from all the vectors in the subspace. Use the following upper bound to prove that the greedy algorithm always succeeds:

$$\sum_{i=0}^{\delta n} \binom{n}{i} \leq 2^{nH(\delta)} .$$

Problem 3 (Cauchy determinant)**.** Let $\mathbb{F}$ be a field containing at least $2n$ distinct elements denoted by $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. Let $A \in \mathbb{F}^{n\times n}$ be a Cauchy matrix: $A_{ij} = \frac{1}{(x_i - y_j)}$. Prove that

$$\det(A) = \frac{\prod_{1\leq i<j\leq n}(x_j - x_i)(y_i - y_j)}{\prod_{1\leq i,j\leq n}(x_i - y_j)} .$$

Conclude that $\det(A) \neq 0$.

Problem 4 (Hadamard is not rigid for high rank)**.** Let $N = 2^n$, and $H_N \in \mathbb{R}^{N\times N}$ be the Walsh-Hadamard matrix defined as follows.

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} ,$$

$$H_N = \begin{pmatrix} H_{N/2} & H_{N/2} \\ H_{N/2} & -H_{N/2} \end{pmatrix} .$$

In particular, $H_N = H_2^{\otimes n}$, where $\otimes$ denotes the Kronecker product.

In this exercise, we will prove that H_N has low rigidity for rank $r \geq N/2$. Namely, $\mathcal{R}^{\mathbb{R}}_{H_N}(N/2) \leq N$.

- Let $A \in \mathbb{R}^{N\times N}$ have eigenvalues $\lambda_1, \ldots, \lambda_N$. Find the eigenvalues of $A - c \cdot I_N$ for $c \in \mathbb{R}$.
- Prove that if $A \in \mathbb{R}^{N\times N}$ has an eigenvalue of multiplicity k, then
$$\mathcal{R}^{\mathbb{R}}_{A}(N-k) \leq N .$$
- Finally, prove that
$$\mathcal{R}^{\mathbb{R}}_{H_N}(N/2) \leq N .$$

Problem 5 (Matrix Norms)**.** Let $M \in \mathbb{C}^{m\times n}$ be a matrix, $k = \min(m, n)$, and $r = \operatorname{rank}(M)$. Let

$$\sigma_1(M) \geq \ldots \geq \sigma_r(M) > \sigma_{r+1}(M) = \ldots = \sigma_k(M) = 0$$

be the singular values of M. Let $\|M\|_F = \left(\sum_{i=1}^m \sum_{j=1}^n |M_{i,j}|^2\right)^{1/2}$ and $\|M\|_2 = \sup_{x\neq 0} \frac{\|Mx\|_2}{\|x\|_2}$ be the Frobenius and spectral norms of M. Prove that

- the Frobenius norm is the root sum of squares of the singular values: $\|M\|_F = \left(\sum_{i=1}^k \sigma_i^2(M)\right)^{1/2}$;
- the spectral norm is the largest singular value: $\|M\|_2 = \sigma_1(M)$;
- if M' is a submatrix of M, then $\sigma_i(M') \leq \sigma_i(M)$. In particular, $\|M'\|_2 \leq \|M\|_2$.